Analytic Solution for Second-Harmonic Gyroresonant Absorption and Mode Conversion

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An entirely analytic treatment for second-harmonic mode conversion is presented, which uses the wave-phase-space (**x**-**k** space) method. Explicit expressions are derived for the coefficients of transmission, reflection, conversion (magnetosonic wave to ion-Bernstein wave), and absorption. A conservation law for wave energy flux in phase space is also presented.

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One of the outstanding analytic problems in the ICRF (ion-cyclotron-range-of-frequencies) heating to tokamaks is the linear mode-conversion process at the ion gyroresonance layer. The complexity of the problem is largely due to the presence of several waves, i.e., the incident and reflected magnetosonic waves, the mode-converted ion-Bernstein wave, and a continuum of Case–van Kampen (CvK) modes (which represent the dissipative gyroresonant absorption by ions). Some approaches to the modeling of this problem[1] have led to a fourth- or or higher-order differential equation that defies analytical solution, thus requiring numerical analysis.

On the other hand, if we consider the problem from the wave-phase-space (**x**-**k** space) point of view, we find that the waves are separated by their characteristic ray paths. Typically they meet only pairwise,[2] at the places where mode conversions occur (Fig. 1). The process of ICRF heating can thus be treated[3] as a succession of pairwise conversions. This recognition has led to an order-reduction scheme,[4] which reduces the fourth-order equation to two second-order equations; in the case of high-field incidence, when the reflected magnetosonic wave is absent, a closed-form analytic solution can be obtained.[5] (The papers referenced above have treated both minority fundamental and majority second-harmonic gyroresonance. In the interest of clarity, we limit our treatment here to the latter mechanism only.) The analysis is now further simplified by the introduction of the concept of *pressure-anisotropy waves*,[6] which travel mainly in **k** space and link the successive mode conversions. In phase space, to model each mode conversion requires only a single *first-order* differential equation, which can be solved analytically.[7] In this Letter we exploit these ideas to derive explicit expressions for the coefficients of transmission, reflection, conversion, and absorption for this problem.

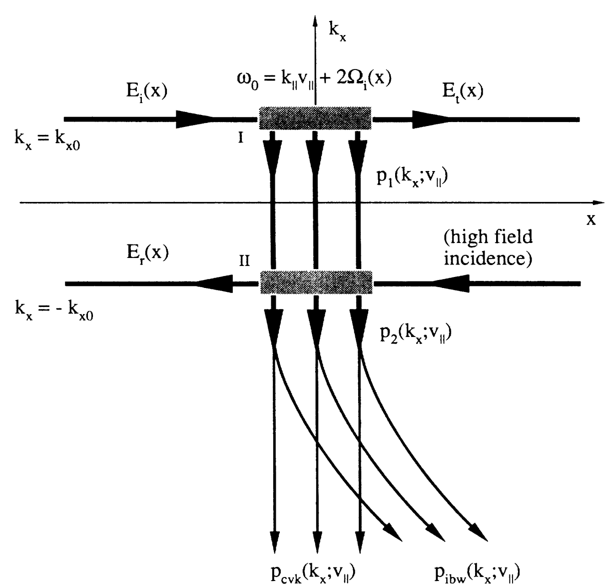

FIG. 1. Schematic diagram of the mode-conversion processes. The magnetosonic waves travel only in x space, while the pressure waves travel only in k_x space. The ion-Bernstein wave travels out of the resonance layer and is absorbed by electron Landau damping.

We consider the standard one-dimensional slab model, with uniform density and temperature, and nonuniform magnetic field $\mathbf{B}_0(x) = B_0(1+x/L_0)\hat{\mathbf{z}}$. The incident magnetosonic wave has a single frequency ω_0, with fixed $k_y, k_z \neq 0$ (Fig. 2). The derivation of the equations is outlined as follows. Consider the linearized Vlasov-Maxwell theory in the guiding-center representation. Denote the guiding-center variables by $(\mathbf{R}, v_{\|}, \mu, \theta_g)$, where $\mathbf{R}$ is the

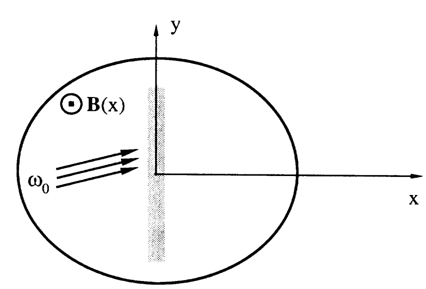

FIG. 2. Schematic diagram of the slab model of a tokamak. The incident wave has frequency ω_0 and fixed $k_y, k_z \neq 0$. The resonance layer is located at $x \approx 0$.

guiding-center position, $v_\parallel$ its parallel velocity, $\mu \equiv m_i v_\perp^2/2B$ its magnetic moment, and θ_g its gyrophase. If the wave has the form $\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t)$, in the eikonal sense, then the linear perturbation f of the guiding-center distribution is a sum $\sum_l f_l \exp i(\mathbf{k}\cdot\mathbf{R}-\omega t+l\theta_g)$, with coefficients f_l containing Bessel functions. The rapidly varying exponentials can be transformed to higher order by the Lie transform technique,[8] or by solving for the perturbed distribution in terms of the electric field, obtaining the current. This familiar procedure of deriving the dispersion tensor for waves[9] introduces resonance denominators. For ions in the second-harmonic resonance layer,

$$\frac{d}{dt}(\mathbf{k}\cdot\mathbf{R}-\omega t+2\theta_g)=\mathbf{k}\cdot\dot{\mathbf{R}}-\omega+2\Omega_i\approx 0,$$

the corresponding exponential is *not* rapidly varying. Our strategy is to *omit* the $l=2$ term from the dielectric tensor; that term then appears in the wave equation as an external current. Thus we obtain two coupled equations, the linearized Vlasov equation for the resonant ($l=2$) particles, and the wave equation driven by their current. All the nonresonant contributions go into the dielectric function in the wave equation. In the k_x representation, we obtain, after extensive but straightforward algebra, which makes use of congruent reduction,[10]

$$\begin{aligned}&\int dk_x'\,\bar{D}_E(k_x,k_x')E(k_x')=\alpha(k_x)\int dv_\parallel p(k_x;v_\parallel),\\ &\int dk_x'\,\bar{D}_p(k_x,k_x';v_\parallel)p(k_x';v_\parallel)=\alpha^*(k_x)E(k_x),\end{aligned}\tag{1}$$

where $E(k_x)$ is the electric-field component of the magnetosonic wave that rotates with the ions (the *resonating component*) and $p(k_x;v_\parallel)\equiv\int d\mu\,\mu f_2(k_x;v_\parallel,\mu)$ is essentially the (v_xv_y) hybrid moment of the perturbed function (the *pressure-anisotropy wave*[6]). $\bar{D}_E$ and $\bar{D}_p$ are two-point dispersion functions, whose Weyl symbols

$$D(k_x,x)\equiv\int\frac{d\sigma}{2\pi}\bar{D}(k_x+\tfrac{1}{2}\sigma,k_x-\tfrac{1}{2}\sigma)\exp(i\sigma x)$$

are local dispersion functions. Let the unperturbed ion distribution be Maxwellian; then to order $(k_\perp v_i/\Omega_i)^2$ we have $\alpha(k_x)=-i(k_x^2+k_y^2)^{1/2}$, and

$$D_E(x,k_x)=\frac{8\omega_i^2(1+3N_\parallel^2)(k_0^2-k_x^2-k_y^2)}{3(k_x^2+k_y^2)-2k_0^2(1-3N_\parallel^2)/N_{\perp 0}^2|},$$

$$D_p(x,k_x;v_\parallel)=\frac{\omega-k_\parallel v_\parallel-2\Omega_i(x)}{c^3(v_i^2/c_A^2)g(v_\parallel)},$$

with $k_0=N_{\perp 0}\omega_0/c_A$, $N_{\perp 0}^2=(1+N_\parallel^2)(1-3N_\parallel^2)/(1+3N_\parallel^2)$, $N_\parallel=k_\parallel c_A/\omega_0$, and $g(v_\parallel)=\exp(-v_\parallel^2/2v_i^2)/(2\pi)^{1/2}v_i$ the $v_\parallel$ distribution. We can ignore the x dependence of D_E, and D_p is k_x independent. Equations (1) can be rewritten as ordinary differential equations:

$$D_E(k_x)E(\cdot)=\alpha(k_x)\int dv_\parallel p(\cdot;v_\parallel),\tag{2a}$$

$$D_p(x;v_\parallel)p(\cdot;v_\parallel)=\alpha^*(k_x)E(\cdot),\tag{2b}$$

where, depending on the representation, the unspecified independent variable is either x or k_x, with the corresponding operator $k_x\to -id/dx$ or $x\to id/dk_x$. These equations have two mode-conversion regions (i.e., $D_E=D_p=0$) at $k_x=\pm k_{x0}\equiv\pm(k_0^2-k_y^2)^{1/2}$, $x=x(v_\parallel)\equiv -k_\parallel v_\parallel/2\Omega_i'$ (Fig. 1). When the incident wave $E_i(x)$ traverses the resonance layer at $x\approx 0$, part of its energy gets converted into $p(k_x;v_\parallel)$, which is governed by the dispersion relation $\omega=k_\parallel v_\parallel+2\Omega_i(x)$, and thus travels only in k_x space at velocity $\dot{k}_x=-\partial\omega/\partial x=-2\Omega_i'$. When this pressure wave crosses the second mode-conversion region at $k_x=-k_{x0}$, part of *its* energy is converted into the reflected wave $E_r(x)$. It is obvious from Fig. 1 that if the incident wave comes from the high-field side no reflection occurs, because the pressure wave cannot go up to the other mode-conversion region. As $p(k_x;v_\parallel)$ is kinetic in $v_\parallel$, it contains the ion-Bernstein wave $p_{\rm IBW}(k_x;v_\parallel)$ as well as the CvK modes $p_{\rm CvK}(k_x;v_\parallel)$. The ion-Bernstein wave is weakly damped and propagates out of the resonance layer, while the CvK modes phase mix and represent gyroresonant absorption. The projection of $p(k_x;v_\parallel)$ to the ion-Bernstein wave is found by the *spectral deformation* technique.[11]

Using the rules of Weyl-symbol calculus,[12] we can derive from Eqs. (1) a conservation law for the wave energy flux in phase space:

$$\frac{\partial}{\partial x}\left[\dot{x}\frac{\partial D_E}{\partial\omega}E^2(k_x,x)\right]+\frac{\partial}{\partial k_x}\left[\dot{k}_x\int dv_\parallel\frac{\partial D_p}{\partial\omega}p^2(k_x,x;v_\parallel)\right]=0,$$

where

$$E^2(k_x,x)\equiv\int ds\,E(x+\tfrac{1}{2}s)E^*(x-\tfrac{1}{2}s)\exp(-ik_xs)$$

is the Wigner function of the E field, and similarly

$$p^2(k_x,x;v_\parallel)\equiv\int\frac{d\sigma}{2\pi}p(k_x+\tfrac{1}{2}\sigma;v_\parallel)p^*(k_x=\tfrac{1}{2}\sigma;v_\parallel)\exp(i\sigma x).$$

The integral form of the conservation law is obtained by our integrating over a box in phase space (see Fig. 1):

$$|\dot{x}|\frac{\partial D_E}{\partial\omega}2\pi|E_i|^2=|\dot{x}|\frac{\partial D_E}{\partial\omega}2\pi|E_t|^2+|\dot{x}|\frac{\partial D_E}{\partial\omega}2\pi|E_r|^2+|\dot{k}_x|\int dv_\parallel\frac{\partial D_p}{\partial\omega}|p_{\rm IBW}|^2+|\dot{k}_x|\int dv_\parallel\frac{\partial D_p}{\partial\omega}|p_{\rm CvK}|^2,\tag{3}$$

where the cross terms of $p_{\rm IBW}$ and $p_{\rm CvK}$ vanish because of the orthogonality of the eigenfunctions in the spectral deformation expansion.

We proceed to the solution. Let $\omega\to\omega_0+i0^+$, simulating the condition that $E(x)$ vanishes as $t\to-\infty$. We use the

x representation in region I; linearize the dispersion function $D_E(k_x)$ about k_{x0},

$$D_E(k_x) \approx -(k_x - k_{x0}) 4c^3 k_{x0}(1+N_\parallel^2)^2/(k_0 c_A N_{\perp 0}^3),$$

with $k_x \to -id/dx$; and evaluate the coupling coefficient α at $k_x = k_{x0}$. Eliminating $p(x;v_\parallel)$ from Eqs. (2) we get a *first*-order ordinary differential equation for $E(x)$, whose solution is

$$E(x) = E_i \exp\left[ik_{x0}x + i\frac{k_0^2 L_0 N_{\perp 0}^4}{4k_{x0}(1+N_\parallel^2)^2}\frac{v_i^2}{c_A^2}\int_{-\infty}^{x} dx' \int dv_\parallel \frac{g(v_\parallel)}{x' - x(v_\parallel) - i0^+}\right], \tag{4}$$

where $E_i = \lim_{x\to -\infty} E(x)\exp(ik_{x0}x)$. Thus we find the transmission coefficient $T(\eta)$:

$$T \equiv \frac{|E_t|^2}{|E_i|^2} = \exp(-2\eta), \text{ where } \eta \equiv \frac{\pi k_0 L_0}{4}\left(\frac{k_0}{k_{x0}}\right)\left(\frac{v_i^2}{c_A^2}\right)\frac{N_{\perp 0}^4}{(1+N_\parallel^2)^2}. \tag{5}$$

This result has been derived previously by many authors. From (4) and (2b), we then find $p(x;v_\parallel)$, Fourier transform it to k_x space, and obtain the pressure wave just after the first mode conversion (at $k_x \lesssim k_{x0}$):

$$p_1(k_x;v_\parallel) = -E_i \frac{2\pi c^3 k_0 L_0}{\omega_0}\frac{v_i^2}{c_A^2} g(v_\parallel)\exp\left[-i(k_x - k_{x0})x(v_\parallel) - \frac{i\eta}{\pi}\int_{-\infty}^{v_\parallel/\sqrt{2}v_i} d\zeta Z(\zeta - i0^+)\right], \tag{6}$$

where

$$Z(\zeta) \equiv \pi^{-1/2}\int_{-\infty}^{+\infty} dx \exp(-x^2)/(x-\zeta)$$

is the plasma dispersion function. Between the two mode-conversion regions, we can ignore the effect of coupling, since it is proportional to $k_\perp^2$. As a result the pressure wave travels in k_x space according to

$$[\omega - k_\parallel v_\parallel - 2\Omega_i(x \to id/dk_x)]p(k_x, v_\parallel) = 0,$$

where $2\Omega_i(x) = \omega_0(1 + x/L_0)$. Since (6) satisfies this equation, it remains valid up to the second mode-conversion region. When the pressure wave crosses region II, it excites the reflected magnetosonic wave $E_r(x)$. Using the k_x representation, and eliminating $E(k_x)$ by (2a), we get a first-order ordinary differential equation for $p(k_x;v_\parallel)$, with solution

$$p(k_x;v_\parallel) = p_1(-k_{x0};v_\parallel)\exp[-i(k_x - k_{x0})x(v_\parallel)] + \theta(k_x + k_{x0})\frac{2\eta}{2+\eta} g(v_\parallel)\exp[-i(k_x + k_{x0})x(v_\parallel)]$$
$$\times \int dv_\parallel' p_1(-k_{x0};v_\parallel')\exp[i2k_{x0}x(v_\parallel')], \tag{7}$$

where $\theta(k_x) = 0, -\frac{1}{2}, -1$ for $k_x > 0, = 0, < 0$, respectively. To find the reflection coefficient we need to know $E_r(x \to -\infty)$. Again using the x representation, we linearize $D_E(k_x)$ about $-k_{x0}$:

$$D_E(k_x) \approx (k_x + k_{x0}) 4k_{x0} c^3 (1+N_\parallel^2)^2/(k_0 c_A N_{\perp 0}^3),$$

with $k_x \to -id/dx$; Eq. (2a) becomes

$$\left(k_{x0} - i\frac{d}{dx}\right)E_r(x) = \frac{-ik_0^2 c_A N_{\perp 0}^3}{4k_{x0}c^3(1+N_\parallel^2)^2}\int dv_\parallel p(x;v_\parallel).$$

Integrating this equation, with the boundary condition $E_r(x\to -\infty) = 0$, we have

$$E_r(x \to -\infty) = -i\frac{k_0 N_{\perp 0}^4}{8k_{x0}c^2(1+N_\parallel^2)^2}\int dv_\parallel p(-k_{x0};v_\parallel).$$

Using Eq. (7) we find, after some algebra, the reflection coefficient R:

$$R(\eta,\kappa) \equiv \frac{|E_r|^2}{|E_i|^2} = \left(\frac{4\eta}{2+\eta}\right)^2 \left|\int_{-\infty}^{+\infty}\frac{d\tau}{\sqrt{\pi}}\exp\left[-\tau^2 + i2\kappa\tau - \frac{i\eta}{\pi}\int_{-\infty}^{\tau} d\zeta Z(\zeta - i0^+)\right]\right|^2, \tag{8}$$

where $\kappa \equiv \sqrt{2}k_{x0}k_\parallel v_i/2\Omega_i'$.

Below region II, i.e., $k_x \lesssim -k_{x0}$, the coupling is no longer negligible. Therefore the equation governing the propagation of the pressure wave here is

$$\left(i\frac{d}{dk_x} + x(v_\parallel)\right)p_2(k_x;v_\parallel) = \gamma(k_x)g(v_\parallel)\int dv_\parallel' p_2(k_x;v_\parallel'), \tag{9}$$

where $\gamma(k_x) = c^3 k_x^2 L_0 v_i^2/[c_A^2 \omega_0 D_E(k_x)]$. By analogy to the CvK analysis of Bateman and Kruskal,[13] we introduce the operator $\hat{L}$, defined by

$$(\hat{L}q)(k_x;v_\parallel) \equiv x(v_\parallel)q(k_x;v_\parallel) - \gamma(k_x)g(v_\parallel)\int dv_\parallel' q(k_x;v_\parallel'),$$

where k_x appears as a parameter. The corresponding eigenvalue problem is $\hat{L}q_W = Wq_W$, and the solution of (9) can be expressed as

$$p(k_x;v_\parallel) = \sum_j c_j(k_x) q_j(k_x;v_\parallel) \exp\left[i \int^{k_x} W_j(k_x') dk_x' \right] + \int dW c_W(k_x) q_W(k_x;v_\parallel) \exp(ik_x W) .$$

However, the eigenvalues W are only the real axis; the discrete ion-Bernstein eigenvalue W_{IBW} is on another Riemann sheet. To expose it we use the spectral deformation method of Crawford and Hislop,[11] which preserves the completeness and orthogonality of the CvK eigenfunctions. The result of this analysis is, in the limit of weak damping $(k_\parallel/k_\perp)^2 \ll 1$,

$$p_{\mathrm{IBW}}(k_x;v_\parallel) = g(v_\parallel) \int dv_\parallel' p(k_x;v_\parallel'), \quad p_{\mathrm{CvK}}(k_x;v_\parallel) \equiv p(k_x;v_\parallel) - p_{\mathrm{IBW}}(k_x;v_\parallel) . \tag{10}$$

Therefore, we can derive the conversion and absorption coefficients C and A, which are defined as the ratios of the wave energy fluxes [see Eq. (3)]:

$$C(\eta,\kappa) \equiv \frac{|\dot{k}_x| \int dv_\parallel (\partial D_p/\partial\omega) |p_{\mathrm{IBW}}|^2}{|\dot{x}| (\partial D_E/\partial\omega) 2\pi |E_i|^2} = \frac{(2-\eta)^2}{8\eta} R(\eta,\kappa) , \tag{11}$$

$$A(\eta,\kappa) \equiv \frac{|\dot{k}_x| \int dv_\parallel (\partial D_p/\partial\omega) |p_{\mathrm{CvK}}|^2}{|\dot{x}| (\partial D_E/\partial\omega) 2\pi |E_i|^2} = 1 - T(\eta) - \frac{(2+\eta)^2}{8\eta} R(\eta,\kappa) . \tag{12}$$

It is easy to check that $T+R+C+A=1$, showing that our approximations are consistent. These coefficients are plotted in Fig. 3; they are in excellent agreement with those of Fuchs and Bers.[5]

In conclusion, we see that **x**-**k** space is the natural setting for studying wave propagation and interaction. By considering the problem in wave phase space, we not only can solve it analytically, but we also gain a better understanding of the physical processes. This method can be adapted to treat minority heating, as well as electron gyroresonant heating (which requires relativistic guiding-center theory[14]). It can also be extended to higher dimensions.[7]

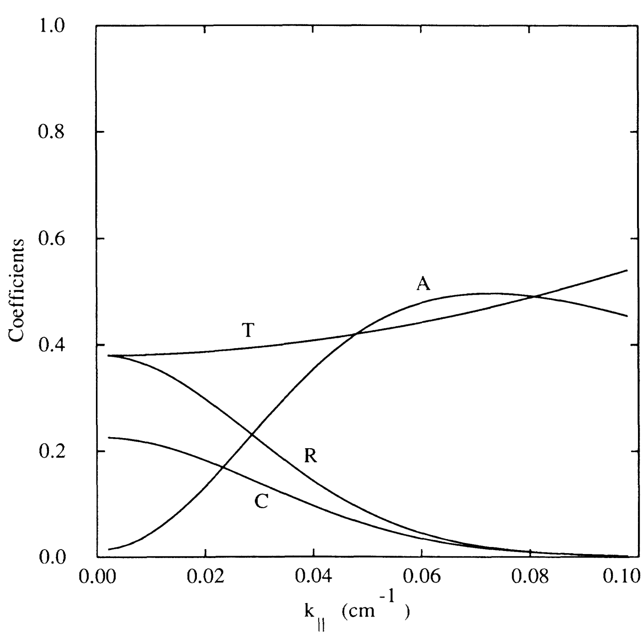

FIG. 3. Coefficients T, R, C, and A as functions of $k_\parallel$, for hydrogen second-harmonic resonance, with $k_y = 0$, $B_0 = 1.4$ T, $L_0 = 132$ cm, $f_0 = 42$ MHz, $n_0 = 4\times10^{16}$ cm^{-3}, and $T_i = 2$ keV. Note that both η and κ vary with $k_\parallel$.

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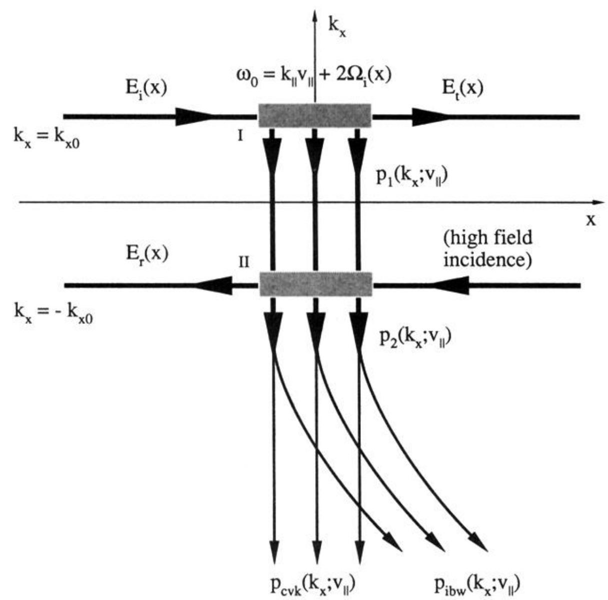

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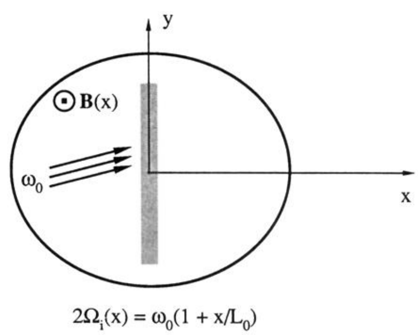

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