Classical-Quantum Correspondence in the Presence of Global Chaos

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We compare the classical and quantum mechanics of a system whose classical dynamics are dominated by chaos. We observe a quantum "scars" of short periodic orbits. By examining the states on a quantum surface of section, we find that the stable and unstable manifolds of the periodic orbits have a clear effect on the eigenstates, in addition to the orbits themselves. A local density operator, constructed from a band of energy eigenstates, is strongly scarred by short periodic orbits. We also demonstrate a relation between the quantum eigenvalue spectrum and the actions of the periodic orbits.

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There has been considerable interest and effort directed toward understanding the quantum-mechanical analog of classical chaotic dynamics.[1] For quasiperiodic systems, the connection between classical and quantum mechanics is made via the Einstein-Brillouin-Keller quantization conditions for classical invariant tori.[2] The development of a semiclassical periodic orbit (PO) theory by Gutzwiller[3] was a major advance towards the goal of a general approach, applicable to both regular and chaotic systems. In the PO theory, the connection between classical and quantum mechanics is made through classical periodic orbits, which appear in the trace of the semiclassical energy Green's function. In principle, this approach allows the calculation of semiclassical eigenvalues for chaotic systems, where the absence of invariant tori precludes Einstein-Brillouin-Keller quantization.

In contrast to properties of the energy spectrum, relatively few formal results addressing the influence of classical mechanics on the *eigenstates* of chaotic systems have been obtained.[4-6] A number of numerical studies have investigated the structure of chaotic eigenstates,[7-11] including the effect of cantori and broken separatrices on the quantum dynamics of driven anharmonic systems,[7] studies of quasienergy eigenstates of the standard map,[8] and investigation of chaotic states in coupled oscillator systems.[9,10] Heller has shown that wave functions of the stadium often have a large probability density in configuration space along certain unstable periodic orbits ("scars").[11]

In this Letter, we study the quantum eigenstates of a globally chaotic system. We observe the effect of periodic orbits in the form of configuration space scars, similar to those described by Heller[11] for the stadium. By using a quantum surface of section (QSOS)[7-10] based on the harmonic oscillator coherent state representation,[12] we show that an even more convincing and unambiguous picture of the connection between unstable periodic orbits can be obtained. We find that the stable and unstable manifolds of the orbits play an important role in the phase-space structure of the eigenstates, in addition to the orbit itself. We also examine the effect of periodic orbits on the phase-space structure of a local density operator, composed of a sum of contiguous energy eigenstates. Earlier investigations of the effect of classical chaos on quantum eigenstates have focused on systems with considerable persistent phase-space regularity[7-10] (see, in particular, the important work of Davis[10]). An exception is Heller's work on the stadium, a completely chaotic system with no invariant tori. However, the stadium is a special system which possesses long-time correlations due to arbitrarily long regular segments of the chaotic trajectories.[13] No previous study of the influence of classical mechanics on eigenfunctions has treated a system as chaotic as the one considered here.

We study the classical and quantum mechanics of the Hamiltonian[14]

$$H=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}x^2y^2+\frac{1-\alpha}{12}(x^4+y^4)\,, \qquad (1)$$

where $\alpha=0.95$. This Hamiltonian exhibits mechanical similarity: The properties of the classical motion at any energy can be determined by scaling from $E=1$. There are thus no complications due to the dependence of phase-space structure on energy.

For $\alpha=0.95$, we have characterized the classical dynamics at $E=1$. The phase space consists almost completely of a single stochastic region. We have calculated a number of periodic orbits of (1). We find two stable periodic orbits, each lying along the coordinate axes. These orbits have residues[15] of 0.996 (i.e., are barely stable), and surrounded by very small regions of quasiperiodicity, which cannot support an Einstein-Brillouin-Keller eigenstate and are barely visible on a surface of section. All other periodic orbits of (1) we found are unstable. The lowest-period orbits lie on the diagonals $y=\pm x$. Each of these "diagonal" orbits has a residue of approximately −59. In addition to locating these and other periodic orbits, we determined the associated invariant stable and unstable manifolds.

We have calculated the quantum eigenvalue spectrum and many eigenstates of (1) by diagonalization in a

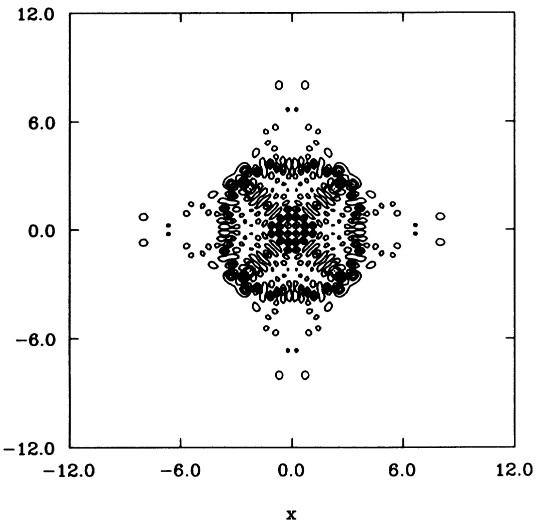

FIG. 1. The configuration-space probability density of the 77th *OOE* state. The energy of this state is 49.468.

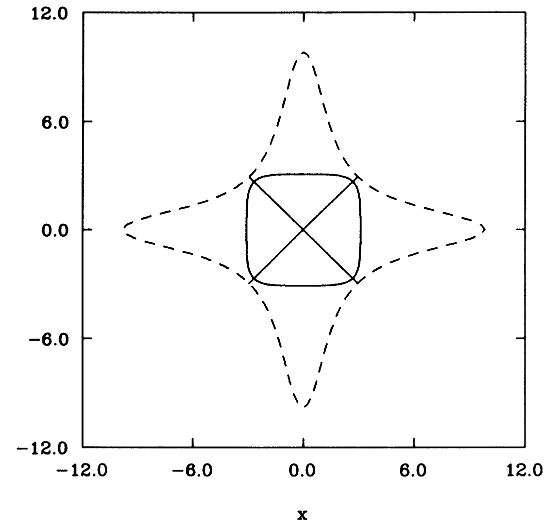

FIG. 2. The periodic orbits that influence the 77th *OOE* state and the $E=49.468$ energy contour.

two-dimensional harmonic oscillator basis.[16] The accuracy of the results was verified by comparing with an independent calculation of eigenvalues and eigenstates by the Feit-Fleck method.[17] To investigate the effect of classical phase-space structure on the eigenfunctions, we plot configuration space probability densities. In addition, we compute QSOS,[7-10] defined as the squared overlap of eigenfunctions with a harmonic oscillator coherent state,[12] centered at the point on the classical surface of section (x,p_x) with $y=0$, and $p_y>0$ determined by energy conservation.

Many of the quantum states are strongly peaked on classical periodic orbits. Figure 1 shows the configuration space $|\psi|^2$ for such a state, which has high density along the diagonals $y=\pm x$ and also around a box-shaped curve. Figure 2 shows the corresponding periodic orbits, along with an equipotential contour. The quantal probability density is enhanced along the classical periodic orbit.

An even more striking correlation is seen in phase space, on the surface of section. Figure 3 shows a contour plot of the QSOS of the same state, together with the corresponding periodic orbits (points) and segments of their stable and unstable manifolds (thick solid curves). The QSOS is dominated by large peaks centered directly on the periodic orbits. Away from the periodic orbits, the probability density remains high along the stable and unstable invariant manifolds. This added structural element is not apparent in the configuration-space plot, which illustrates the utility of the QSOS representation. The state in Fig. 3 is not associated with a separatrix between distinct types of classical motion (see Ref. 10 for examples of this type of state); it is influenced by the stable and unstable manifolds of an isolated unstable fixed point, which is itself embedded in a sea of chaos.

For this particular state, we find that the actions of the scarring orbits are close to quantizing values. Although this result is suggestive, we have not found such a rule to reliably predict which states are scarred by a given periodic orbit.

The QSOS of many eigenstates of (1) are strongly peaked on the diagonal box, and other known orbits. A significant fraction, however, are not obviously related to any classical periodic trajectories. These may be related to long period or highly unstable motion, which we have been unable to locate, or they may simply be unrelated to the classical motion.

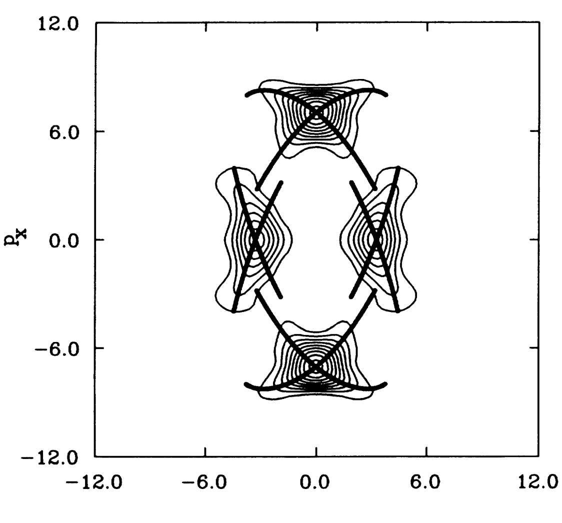

FIG. 3. Contours of the QSOS of the 77th *OOE* state superimposed on the separatrix manifolds of the "diagonal" and "box" orbits.

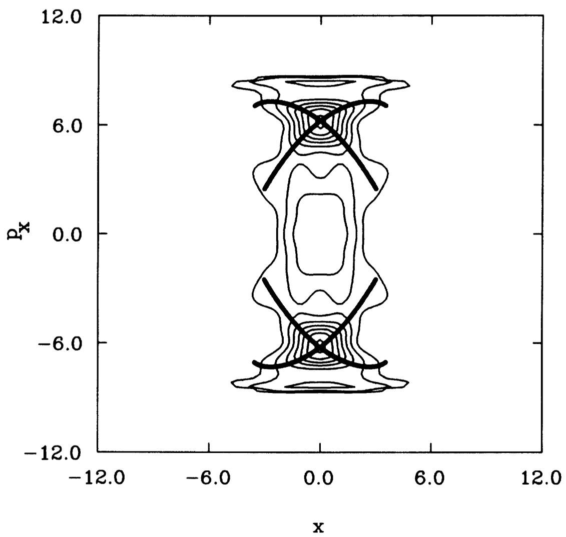

FIG. 4. The QSOS of the LDO for all eigenstates in the range (37.571,40.096). The thick dark lines are the separatrix manifolds of the diagonal orbits.

The PO theory of energy spectra relates the entire manifold of eigenvalues to a complete sum over all orbits,[3] and thus does not imply any relation between individual eigenstates and particular classical trajectories. However, the work of Du and Delos[4] suggests a general connection between periodic orbits with short periods and quantum mechanical quantities which are averaged over a correspondingly broadened range of energies. In this spirit, we examine the behavior of groups of nearby eigenstates, defined by a local-density operator (LDO)

$$\rho = \sum_{k=k_{\min}}^{k_{\max}} |\phi_k\rangle\langle\phi_k| , \tag{2}$$

where the sum includes all states in a selected energy range. The QSOS of the LDO is simply the sum of the QSOS of the individual states. We find that it is very strongly influenced by short-period orbits. A typical case is shown in Fig. 4. Here, a sum of 42 eigenstates with mean energy $E=38.83$ has been performed. If the individual eigenstates are randomly peaked, the LDO should tend to be featureless. Density arising from fluctuations not associated with the organizing classical phase-space structure should lead to a smooth and uniform QSOS, while recurrent quantum structure associated with orbits and invariant manifolds should lead to a buildup of density. This is indeed what we observe: The QSOS is sharply peaked on the diagonal periodic orbits and their associated manifolds. The uninterpretable structure of many of the individual eigenstates has been smoothed to give a featureless background. It should be noted that only 4 of the 42 states in the sum were themselves localized on the diagonal orbits. We have studied a number of different energy regimes, and in all cases, the LDO QSOS is very strongly correlated with the short-period orbits. Very recently, Bogomolny introduced a theory of scarred eigenstates which predicts that groups of states within an energy range should be localized near short periodic orbits.[5] This is also suggested by unpublished work of Delos.[6] The present work provides a numerical verification of these important formal results for a strongly chaotic system.

The PO theory predicts that the semiclassical density of states

$$n(E) = \sum_k \delta(E - E_k) , \tag{3}$$

where E_k is the kth quantum eigenvalue, can be rewritten as[3]

$$n(E) = \bar{n}(E) + \sum_r \sum_{j=1}^{\infty} a_{rj}(E) \cos\{j[S_r(E) - \mu_r]\} , \tag{4}$$

where r labels the primitive periodic orbits and j labels the repetitions of these primitive orbits. $\bar{n}(E)$ is the average (Thomas-Fermi) density of states, and $S_r(E)$ and μ_r are the action and Maslov index of the rth orbit, respectively. By use of the scaled energy variable $\xi = E^{3/4}$ and the scaling property of the classical actions, $S(E) = E^{3/4} S(1)$, Eqs. (3) and (4) can be written

$$\left[\frac{3}{4} \sum_k \frac{\delta(\xi - \xi_k)}{\xi} \right] - \bar{n}(1) = \sum_r \sum_{j=1}^{\infty} a_{rj}(1) \cos\{j[\xi S_r(1) - \mu_r]\} . \tag{5}$$

By Fourier transformation with respect to ξ, the left-hand side of Eq. (5) becomes

$$\left[\frac{3}{4} \sum_k \frac{e^{i\omega\xi_k}}{\xi_k} \right] - 2\pi\bar{n}(1)\delta(\omega) . \tag{6}$$

The peaks in the Fourier transform of the right-hand side of Eq. (5) occur at integer multiples of the classical actions of periodic orbits at energy $E=1$. To examine the nature and validity of the semiclassical PO expression, we have computed the Fourier transform of the exact scaled eigenvalue of (1), using the first 482 converged levels. This result is shown in Fig. 5, along with the predicted locations of peaks based on the classical actions at $E=1$. A close agreement is clearly apparent; we believe the unexplained peaks are associated with unknown periodic orbits.[18]

To summarize, we observe, for the first time, a clear correspondence between quantum eigenstates and classical periodic orbits of a strongly chaotic system with a smooth potential. We emphasize that we are not seeing the association of quantum density with residual *regular* phase-space structure, such as resonance zones or cantori.[7–10] Rather, we see connections between classical

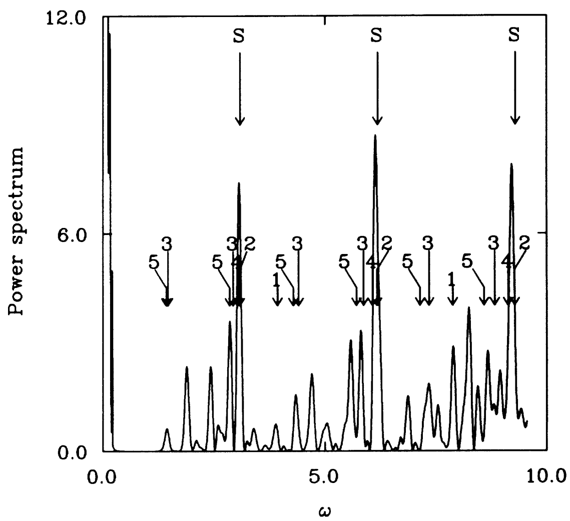

FIG. 5. The power spectrum of the scaled energy spectrum. Arrows mark integer multiples of the scaled action of periodic orbits. Labeling of arrows: S—the stable x- and y-axis orbits; 1—the least unstable orbit; 2—the next least unstable orbit; etc.

and quantum mechanics in the almost complete absence of such structure. This extends the numerical evidence for scars of periodic orbits beyond the special case of the stadium.[11] The effect of short periodic orbits is even more apparent in the LDO, a result which is consistent with very recent formal results of Bogomolny[5] and Delos.[6] We anticipate that the numerical results obtained here will stimulate further analytic developments.

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