Wormholes and Goldstone Bosons

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The quantum theory of a complex scalar field coupled to gravity is considered. A formalism for the semiclassical approach in Euclidean time is developed and used to study wormhole physics. The conserved global charge plays an essential role. Wormhole physics turns on only after the symmetry is spontaneously broken. An effective self-interaction for Goldstone bosons due to wormholes is shown to be a cosine potential, whose vacuum energy will be reduced by the cosmic expansion. Some implications and questions are discussed.

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The physics of geometrical fluctuations in quantum gravity has been studied[1-3] to implement the idea[4] that small-scale fluctuations lead to an apparent loss of quantum coherence on a large scale. However, it has been shown[5,6] that there is no loss of quantum coherence due to geometrical fluctuations such as wormholes. Coleman[7] has proposed a solution to the cosmological-constant problem by employing wormhole physics.

Giddings and Strominger[3] have found a wormhole solution as an instanton configuration of Euclidean quantum gravity coupled to a massless axion, which arises from the antisymmetric rank-three tensor in the superstring theories. Also, they wrote down an effective Hamiltonian for the interaction between wormholes and the K_0-$\bar{K}_0$ system.[6]

In this Letter, a general formalism for tunneling or instanton physics in Euclidean time is discussed in theory of a complex scalar field. This will show that wormholes exist in a more general frame. By use of this, the properties of wormhole solution are studied. The wormhole solution exists only when the symmetry is spontaneously broken. An effective interaction on Goldstone bosons by wormhole physics is obtained and is shown to exhibit the cosine potential exactly as in the case of the ordinary axion interacting with QCD instantons.[8,9] With a non-Abelian symmetry, the effective potential is not well defined. Finally, some discussion will follow.

Consider a quantum-mechanical system of a particle of unit mass which is moving on a plane with a potential $V(r)$ and angular momentum $Q=r^2\dot{\theta}$. The energy is $E=\dot{r}^2/2+V_{\rm eff}(r)$, where $V_{\rm eff}(r)=Q^2/2r^2+V(r)$. Suppose that the particle is at a metastable local minimum of $V_{\rm eff}$. The standard semiclassical approach leads to tunneling rate proportional to $\exp[-2\int dr(2V_{\rm eff})^{1/2}]$, where the rangle of integration is from the metastable point to the escape point.

Now the bounce solution, which gives the maximum rate, can be obtained by solution of the Euclidean equation, $-\ddot{r}-Q^2/r^3+V'(r)=0$, of the Euclidean action $S^E_{\rm eff}=\dot{r}^2/2+V_{\rm eff}(r)$. The exponential factor becomes the exponent of minus the bounce action.

Let us reintroduce the real angle variable θ and so angular momentum becomes $Q=r^2\dot{\theta}$. Euclidean action becomes $S^E=\int d\tau[\dot{r}^2/2+r^2\dot{\theta}^2/2+V(r)]$. The variational principle, $\delta S^E=0$, will yield

$$-\ddot{r}+r\dot{\theta}^2+V'(r)=-\ddot{r}+Q^2/r^3+V'(r)=0$$

for r. One can see easily that this is a wrong equation.

What went wrong? Because angular momentum is conserved by the superselection rule, the tunneling process should satisfy angular momentum conservation. The conservation of angular momentum should be a constraint in the variation of the Euclidean action. (The way to justify these statements in the path-integral method is not known to the author.) The correct variational principle is

$$\delta\left[S_E+\int d\tau\lambda(\tau)\,dQ/d\tau\right]=0, \qquad (1)$$

with a Lagrangean multiplier, λ.

Now I can generalize this formalism easily into quantum field theory of a complex scalar field $\phi\equiv f\exp(i\theta)/\sqrt{2}$. The Euclidean action is

$$S_E=\int d\tau d^3x[\tfrac{1}{2}\dot{f}^2+\tfrac{1}{2}f^2\dot{\theta}^2+V(f)]. \qquad (2)$$

The total charge $Q=\int d^3x f^2\dot{\theta}$ is conserved. As the charge is conserved, the variational principle (1) produces the right bounce equation. But the bounce equation becomes messy because the solution for the Lagrangean multiplier is not local in space.

To overcome this, consider possible tunneling paths, say $\phi(\mathbf{x},\tau)$, which satisfy the initial condition and the charge conservation. However, they do not need to satisfy the local current conservation, $\partial_\mu j^\mu=0$, where $j_\mu=f^2\partial_\mu\theta$. One can divide these possible paths into sets, each of which is characterized by a source $\kappa(x)$ of the current, $\partial_\mu j^\mu=\kappa(x)$. We have $\int d^3x\,\kappa(\mathbf{x},\tau)=0$ for the overall charge conservation.

Let us first find the bounce solution inside a set of possible tunneling paths characterized by a given source

$\kappa(x)$. The variational principle is then

$$\delta\left[S_E+\int d^4x\,\lambda(x)(\partial_\mu j^\mu-\kappa)\right]=0. \tag{3}$$

The equations imply that $\lambda=\theta+\text{const}$. After substitution of this, the equations for f and θ become

$$-\partial_\mu^2 f-f(\partial_\mu\theta)^2+V'=0, \tag{4}$$

$$\partial_\mu(f^2\partial_\mu\theta)=\kappa(x). \tag{5}$$

Note the minus sign of the second term in Eq. (4).

The bounce solution satisfies charge conservation and is a stationary point of the Euclidean action if the variations are constrained to satisfy charge conservation. What the variational principle Eq. (3) means is that by taking a variation in the set of a given source, we have frozen the variation along the source direction and found the stationary solution. Let us call the stationary solution $\phi_\kappa(x)$ for each κ.

Now compare the actions of the stationary solutions for different κ's, say κ and $\kappa+\epsilon$, where ϵ is infinitesimal. $\phi_{\kappa+\epsilon}$ can be written as the sum of ϕ_κ and a variation $\delta\phi_\kappa$. In terms of f and θ, the bounce solutions at κ ($\kappa+\epsilon$) are f,θ ($f+\delta f,\theta+\delta\theta$). By taking variation of Eqs. (4) and (5), we obtain equations for δf and $\delta\theta$,

$$-\partial^2\delta f-\delta f(\partial\theta)^2-2f(\partial\theta)(\partial\delta\theta)+V''\delta f=0, \tag{6}$$

$$2\partial_\mu(f\,\delta f\,\partial_\mu\theta)+\partial_\mu(f^2\partial_\mu\delta\theta)=\epsilon(x). \tag{7}$$

By using Eqs. (6) and (7) and partial integrations one can obtain the difference between the actions,

$$\delta S_\kappa^E=S^E(\phi_{\kappa+\epsilon}^b)-S^E(\phi_\kappa^b)=-\int d^4x\,\epsilon(x)\theta(x). \tag{8}$$

One can expect that there is a solution of the equation, $\partial_\mu(f^2\partial_\mu\Delta\theta)=\epsilon$, where $\Delta\theta$ is order of ϵ. The reason is that when the charge density is not zero, f is not zero. One can thus take perturbation of Eq. (5) in θ and κ. Then Eq. (8) becomes

$$\delta S_\kappa^E=-\int d^4x\,\kappa\Delta\theta, \tag{9}$$

which is nonzero in the first order of ϵ except when $\kappa=0$. For $\kappa=0$, the first variation of the action vanishes. The bounce solution satisfies Eqs. (4) and (5) with $\kappa=0$ and the bounce action gives the exponential suppression factor in the tunneling rate.

Let us couple gravity to a complex scalar field. The Euclidean action is

$$S^E=\int d^4x\sqrt{g}\,[-(M_p^2/16\pi^2)R+\tfrac{1}{2}g^{\mu\nu}\partial_\mu f\partial_\nu f+\tfrac{1}{2}f^2g^{\mu\nu}\partial_\mu\theta\partial_\nu\theta+V(F)]. \tag{10}$$

There is a boundary term, which is not relevant here. There is also a term for the Euler number of the manifold, which will be neglected as it is not essential in the following discussion.

The previous argument for the variational principal goes through here too. The Euclidean solution would satisfy the charge conservation, $\partial_\mu(j^\mu\equiv\sqrt{g}g^{\mu\nu}f^2\partial_\nu\theta)=0$. The variational principle in Eq. (3) with $\kappa=0$ will lead to the rest of the field equations,

$$g^{-1/2}\partial_\mu(\sqrt{g}\partial^\mu f)-f\partial_\mu\theta\partial^\mu\theta+V'(f)=0, \tag{11}$$

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=(8\pi/M_p^2)T_{\mu\nu}, \tag{12}$$

where

$$T_{\mu\nu}=\partial_\mu f\partial_\nu f-f^2\partial_\mu\theta\partial_\nu\theta-g_{\mu\nu}[\tfrac{1}{2}\partial_\sigma f\partial^\sigma f-\tfrac{1}{2}f^2\partial_\sigma\theta\partial^\sigma\theta+V(F)].$$

(The famous minus sign of the Euclidean energy-momentum tensor for the phase has been obtained independently by Giddings and Strominger[3] for the axion field arising from the rank-three antisymmetric tensor of the superstring theories.)

Suppose that the potential has an absolute minimum at $f=v$ with value $V_0=V(v)$. One can choose freely $v\ll M_p$. Even though the symmetry is spontaneously broken, the total charge is still conserved. There is a massless Goldstone boson represented by the field $a(x)=v\theta$. Assume that the solution is O(4) symmetric. The metric is $ds^2=d\rho^2+R(\rho)^2d\Omega_3^2$, where $d\Omega_3^2$ is the line element of a unit three-sphere. The current-conservation equation has a solution

$$\dot\theta(\rho)=n/2\pi^2f(\rho)^2R(\rho)^3, \tag{13}$$

so that $\int_{S^3}(j^0=R^3f^2\dot\theta)=n$, where the integration is over any three-sphere around the origin. The global charge n becomes an integer after quantization. Equations (11) and (12) become

$$-\ddot f-\frac{3\dot R}{R}\dot f-\frac{n^2}{4\pi^4f^3R^6}+V'(f)=0, \tag{14}$$

$$\dot R^2=1-\frac{8\pi}{3M_p^2}\left(\frac{n^2}{8\pi^4f^2R^6}+V_0\right)R^2. \tag{15}$$

If we freeze $f(\rho)$ to be v, we can solve Eq. (15) exactly. The general case will be discussed later. There are two parameters in the solution; the radius r of the wormhole in the flat-space background such that $r^2=n/(3\pi)^{1/2}\pi vM_p$ and the Hubble constant H of the De Sitter space-time such that $H^2=8\pi V_0/3M_p^2$. The solution exists only if $\sqrt{\pi}vM_p^3/4n>V_0$, i.e., the size of the wormhole is smaller than the horizon length. The solution is

$$H\rho = \frac{\beta}{[\alpha(\beta-\gamma)]^{1/2}} \Pi \left[h\left(\frac{R^2}{r^2}\right), \frac{\alpha-\beta}{\alpha}, \left(\frac{\gamma(\beta-\alpha)}{\alpha(\beta-\gamma)}\right)^{1/2} \right], \tag{16}$$

$$h(x) = \arcsin\left[\frac{\alpha(x-\beta)}{x(\alpha-\beta)}\right]^{1/2},$$

where Π is the elliptic integral of the third kind and $\alpha > \beta > 0 > \gamma$ are roots of $x^3 - x^2/(rH)^2 + 1/(rH)^2 = 0$. This is a wormhole solution which connects two opposite points of Euclidean De Sitter space-time. The action of the wormhole solution is

$$S_b = -\frac{3M_p^4}{8V_0}\frac{rH}{[\alpha(\alpha-\gamma)]^{1/2}}\left[\alpha E\left(\frac{\pi}{2},p\right) + \alpha\gamma F\left(\frac{\pi}{2},p\right) - 3\Pi\left(\frac{\pi}{2},\frac{\alpha-\beta}{\alpha},p\right)\right], \tag{17}$$

where $p = [(\alpha-\beta)/(\alpha-\gamma)]^{1/2}$ and E (F) is the elliptic integral of the second (first) kind.

As $V_0 \to \sqrt{\pi} v M_p^3/4n$, the bounce solution collapses into a three-sphere with zero action. When $V_0 \to 0$, $S_b - S_v = (3\pi)^{1/2} M_p/2v$. (The solution in the case $V_0 = 0$ has been given in Ref. 3.) If the symmetry-breaking scale v is much lower than M_p, the action is then much larger than $\hbar$ and the radius of the wormhole becomes larger than the Planck scale, and so the semiclassical approximation is good. The vacuum solution and action are $a = 0$, $R(\rho) = H^{-1}\sin(H\rho)$, and $S_v = -3M_p^4/8V_0$. The contribution of wormholes is then $K\exp[-(S_b - S_v)]$, where K is of order of the size of the wormhole.

The interpretation of the wormhole solution depends on whether or not there is a negative mode around the solution. If there is a negative mode, the solution is called a bounce and describes the nucleation and growth of wormholes in the Minkowski time. If there is no negative mode, the solution is called an instanton and describes the tunneling and mixing of two states of the same energy. (See Ref. 8 for more explanation.) There is a simple but not complete argument that the wormhole solution is not a bounce but an instanton solution in Euclidean time. Consider the asymptotically flat case. As τ goes from $-\infty$ to $+\infty$, the charge of our Universe changes by $\Delta Q = n$ as the sign of the current changes, or that of $d\rho/d\tau$. This will sink into the wormhole, appearing in another side. However, the bounce solution should bounce back to the initial configuration and there should be no charge difference.

What happens if we make $f(x)$ dynamical rather than take $f(x)$ to be V? A detailed study of Eqs. (14) and (15) shows that at the neck of the wormhole f will have a value smaller than v, but nonzero, and the size of the neck and the action get bigger. As one can see easily, the size and action go to infinity as the symmetry is restored.

What is the effect of the formation of wormholes in Euclidean time on our Universe? That is captured in Refs. 5 and 6 as an effective Hamiltonian after summation over all possible combinations of formation of wormholes in the dilute-gas approximation. When one restricts oneself to wormholes of charge ± 1, with creation (annihilation) operators $a_+^\dagger, a_-^\dagger$ (a_+, a_-), the Hamiltonian density becomes $\mathcal{H}_{\text{eff}} = Ke^{-S}(C + C^\dagger)$, where $C = a_+^\dagger + a_-$, $C^\dagger = a_+ + a_-^\dagger$. Note that $[C, C^\dagger] = 0$. Additionally, C and $C^\dagger$ are independent of space-time as wormholes do not carry any energy momentum.

As the total charge of universes is conserved, the operator $Q - a_+^\dagger a_+ + a_-^\dagger a_-$ will commute with the effective Hamiltonian. This implies that the effective interaction is

$$\mathcal{H}_{\text{eff}} = Ke^{-S}(e^{ia/v}C + e^{-ia/v}C^\dagger), \tag{18}$$

which is invariant under global U(1) rotation, $a/v \to a/v + \epsilon$, $C \to e^{-i\epsilon}C$, and $C^\dagger \to e^{-i\epsilon}C^\dagger$.

After many measurements, one expects that our Universe will settle into one of the eigenstates of C and $C^\dagger$.[5,6] The explicit form of the eigenstate is

$$|\alpha\rangle = \pi^{-1/2}\exp[\alpha\alpha^* - (\alpha - a_+^\dagger)(\alpha^* - a_-^\dagger)|0\rangle,$$

where $C|\alpha\rangle = \alpha|\alpha\rangle$, $C^\dagger|\alpha\rangle = \alpha^*|\alpha\rangle$. Let us put $\alpha = ce^{-i\theta_0}$. The effective Hamiltonian density on this eigenstate becomes

$$\mathcal{H}_{\text{eff}} = cKe^{-S}\cos(a/v - \theta_0). \tag{19}$$

This is the same form as the axion potential due to the QCD instanton effects.[7,8] The Goldstone boson becomes massive through the quantum-gravity effects and the total charge in our Universe is not conserved. The mass of the Goldstone boson, $M_a^2 = (c/v^2)Ke^{-S}$, is now dependent on a parameter c of the wave function of our Universe.

Let me try to include wormholes carrying charge $\pm n$. These are qualitatively different from the sum of n wormholes of charge ± 1 because of topological difference. Hence there is an effective potential for each n similar to Eq. (19) with different K and S. One may not expect any correlations between eigenvalues α_n for different n. If one sums over all n, the Hamiltonian becomes a sum over n of $A_n\cos(na/v - \theta_n)$ with amplitudes and phases independent of each other. In this case

Goldstone boson dynamics becomes more involved.

Suppose that the scalar field is in a representation of a non-Abelian group and that the symmetry is spontaneously broken. There are Goldstone bosons and corresponding wormhole solutions. The effective creation and annihilation operators of wormholes will have commutation relations as operators of the scalar field. Annihilation operators commute with each other and so are creation operators. There is then no problem to write a coherent state for all annihilation operators. The charge operators will, however, satisfy a non-Abelian algebra. The effective interaction between Goldstone bosons and wormholes will be similar to that of the Abelian case.

In this Letter, the wormhole dynamics arising from quantum mechanics of a complex scalar field coupled to gravity has been studied. The wormhole physics will turn on after spontaneous symmetry breaking. The induced effective interaction of Goldstone bosons has a cosine potential, whose parameters depend on the state of our universe. The possible vacuum energy will be red-shifted into nonrelativistic massive Goldstone bosons as the Universe expands.

Suppose that the symmetry breaking occurs later than inflation. Will the wormhole effect turn on immediately? Otherwise, did it happen in the past, or will it in the future? A cosine potential looks bad for the cosmological-constant problem. Will multicharged wormhole effects rescue the situation?

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