The *XY* Model Has Long-Range Order for All Spins and All Dimensions Greater than One

Tom Kennedy,[a] Elliott H. Lieb, and B. Sriram Shastry[b]

Department of Physics, Princeton University, P.O. Box 708, Princeton, New Jersey 08544

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The quantum *XY* model of interacting spins on a hypercubic lattice has long-range order in the ground state for all values of the spin and all dimensions greater than one. We also show that in the limit of high dimension the spontaneous magnetization converges to the spontaneous magnetization of the Néel state.

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We consider the question of the existence of long-range order (LRO) in the ground state of the quantum *XY* model on the hypercubic lattice in two and higher dimensions. (We note that there cannot be any long-range order in two dimensions at positive temperature T, as the Mermin-Wagner-Hohenberg theorem shows.) The Hamiltonian is

$$H = -\sum_{\langle x,y\rangle} (S_x^1 S_y^1 + S_x^2 S_y^2)\,, \tag{1}$$

with $\langle x,y\rangle$ denoting nearest-neighbor pairs. Dyson, Lieb, and Simon (DLS)[1] showed the existence of LRO for spin $S=\frac{1}{2}$ in 3D for positive temperature T. Neves and Perez[2] noticed that one could take the $T\to 0$ limit of the DLS formalism and thereby prove LRO for the ground state of the *XXX* Heisenberg model in two dimensions for $S\geq 1$. Kubo[3] used the methods in Refs. 1 and 2 to show the existence of LRO for $S\geq 1$ in 3D and for $S\geq \frac{3}{2}$ in 2D for the *XY* model in (1).

In a recent paper,[4] we were able to improve on the DLS method and applied this improvement to the *XXX* Heisenberg antiferromagnet to show ground-state LRO for $S=\frac{1}{2}$ in 3D. Here we show that this method is also capable of proving ground-state LRO for the *XY* model (1) for $S\geq\frac{1}{2}$ and all dimensions greater than one. The case $S=\frac{1}{2}$ corresponds to bosons with hard-core repulsion (at half-filling) via the lattice-gas analogy of Matsubara and Matsuda[5] and proves that Bose-Einstein condensation (or off-diagonal long-range order) occurs in this system despite hard-core repulsion in real space. *In fact, these $S=\frac{1}{2}$ systems provide the only examples known to us of interacting particles in which Bose-Einstein condensation has been proved to occur.*

For a finite ν-dimensional hypercubic lattice Λ the ground state is unique.[6] The object of principal interest is the Fourier transform of the two-point function. That is, for $\alpha=1$, 2, or 3 we set

$$\hat{S}_p^\alpha = |\Lambda|^{-1/2} \sum_{x\in\Lambda} S_x^\alpha \exp(ip\cdot x)\,,$$

and then set

$$g_p^\alpha = \langle \hat{S}_p^\alpha \hat{S}_{-p}^\alpha\rangle\,,$$

with the angular brackets denoting ground-state expectation value. By Parseval's identity

$$|\Lambda|^{-1}\textstyle\sum_p g_p^\alpha = \langle (S_0^\alpha)^2\rangle\,,$$

and by taking the usual infinite-volume limit and passing from sums to integrals, we obtain

$$(2\pi)^{-\nu}\int d^\nu p\, g_p^1 = \langle (S_0^1)^2\rangle\,, \tag{2}$$

where the integral is over the cube $|p_i|\leq\pi$ for $i=1,\ldots,\nu$. Note that in (2), and henceforth, g_p^α denotes the infinite-volume limit of g_p^α for finite volume.

Equation (2) will not be used here but instead we follow Ref. 4 and introduce the additional sum rule which is derived in the same way as (2) for $i=1,\ldots,\nu$,

$$(2\pi)^{-\nu}\int d^\nu p\, g_p^1 \cos p_i = \langle S_0^1 S_{\delta_i}^1\rangle \equiv e_1\,, \tag{3}$$

where δ_i is the nearest neighbor to 0 in the direction i. Clearly $\langle S_0^1 S_{\delta_i}^1\rangle$ is independent of i in the infinite-volume limit. By symmetry e_1 is minus half the ground-state energy per bond. (Note that we are treating the ferromagnetic *XY* model, so that $e_1>0$. The antiferromagnetic and ferromagnetic *XY* models are isomorphic.[1]) It might be thought that (3) demands more information (i.e., e_1) than does (2), but this is misleading because our bound on g_p^1 below also requires knowledge of e_1.

The bound on g_p^1 for $p\neq 0$ is

$$0\leq g_p^1 \leq \tfrac{1}{2}\left[\sum_{i=1}^{\nu}(e_1 - e_3\cos p_i)\Big/\sum_{i=1}^{\nu}(1-\cos p_i)\right]^{1/2}, \tag{4}$$

where $e_3=\langle S_0^3 S_{\delta_i}^3\rangle$. This bound is true for every finite Λ and hence remains true in the limit $\Lambda\to\infty$. It appears several times in the literature and so we shall not attempt to outline its lengthy derivation. It can be derived as a $T\to 0$ limit of the bound in DLS[1] as done (in the *XXX* context) in Ref. 2. It can also be derived directly in the ground state, as done in the *XXX* context in Ref. 4. As Kubo points out, $|e_3|\leq e_1$ (for otherwise the energy could be lowered by interchanging the 1 and 3 spin directions), so that

$$0 \le g_p^1 \le \tfrac{1}{2}(e_1)^{1/2}\left[\left(\nu + \left|\sum_{i=1}^{\nu}\cos p_i\right|\right)\Big/\sum_{i=1}^{\nu}(1-\cos p_i)\right]^{1/2} \quad (5)$$

The sum rule (3) says

$$e_1 = (2\pi)^{-\nu}\int d^\nu p\, g_1^p \nu^{-1}\sum_{i=1}^{\nu}\cos p_i . \quad (6)$$

Let $m^2 = \lim_{|x|\to\infty}\langle S_0^1 S_x^1\rangle$ be the square of the spontaneous magnetization in the 1 direction. Our goal, LRO, is equivalent to $m \neq 0$. Since the right-hand side of (5) is integrable, $m \neq 0$ if and only if g_p^1 (in the infinite-volume limit) has a delta function at $p=0$. In fact, $(2\pi)^\nu m^2$ is the coefficient of this delta function. The bounds (5) and the sum rule (6) imply

$$e_1 \le \tfrac{1}{2}(e_1)^{1/2} I(\nu) + m^2 , \quad (7)$$

where

$$I(\nu) = (2\pi)^{-\nu}\int d^\nu p\left[\sum_{i=1}^{\nu}(1+\cos p_i)\Big/\sum_{i=1}^{\nu}(1-\cos p_i)\right]^{1/2}\left\{\nu^{-1}\sum_{i=1}^{\nu}\cos p_i\right\}_+ , \quad (8)$$

and $\{a\}_+ = \max\{a,0\}$. Numerically we find $I(2)=0.65$ and $I(3)=0.35$. A simple variational argument (using the wave function with all spins aligned in the 1 direction) shows $e_1 \ge \frac{1}{2}S^2$ in all dimensions. Thus for any spin $S \ge \frac{1}{2}$ and $\nu=2$ or $\nu=3$ we conclude from (7) that $m \neq 0$, thereby implying there must be LRO in the XY model.

To prove LRO for $\nu > 3$ we must bound $I(\nu)$. We shall now prove that $I(\nu) \le I(2)$ for $\nu \ge 3$, and thereby establish [from (7)] LRO for all $\nu > 3$. The proof is as follows. Let $F(x) = x[(1+x)/(1-x)]^{1/2}$ for $0 \le x \le 1$. One checks that F is monotone increasing and convex. For $i,j = 1,\dots,\nu$ let $Y_{ij}(p) = \frac{1}{2}(\cos p_i + \cos p_j)$ and let $Y(p) = \nu^{-1}\sum_{i=1}^{\nu}\cos p_i$. The integral in (8) is $F(\{Y(p)\}_+)$. Now

$$Y(p) = (\nu^2-\nu)^{-1}\sum_{i\neq j} Y_{ij}(p) .$$

Since $\{a+b\}_+ \le \{a\}_+ + \{b\}_+$, we have

$$\{Y(p)\}_+ \le (\nu^2-\nu)^{-1}\sum_{i\neq j}\{Y_{ij}(p)\}_+ .$$

Since F is monotone,

$$F(\{Y(p)\}_+) \le F\left((\nu^2-\nu)^{-1}\sum_{i\neq j}\{Y(p)\}_+\right) .$$

Since F is convex,

$$F(\{Y(p)\}_+) \le (\nu^2-\nu)^{-1}\sum_{i\neq j}F(\{Y_{ij}(p)\}_+) .$$

But

$$(2\pi)^{-\nu}\int F(\{Y_{ij}(p)\}_+)\,d^\nu p = I(2) .$$

This proves $I(\nu) \le I(2)$. By the same analysis one can also prove that $I(\nu) \le I(\mu)$ whenever $\nu > \mu$.

Having achieved our goal of proving LRO, we now turn to an additional fact about the spontaneous magnetization in the limit of high dimension. We shall show that m^2 converges to the classical value $S^2/2$ as $\nu \to \infty$. First, we prove that $I(\nu) \to 0$ as $\nu \to \infty$. Using $\{Y(p)\}_+ \le |Y(p)| \le 1$ and the Schwarz inequality, we have

$$I(\nu)^2 \le 2(2\pi)^{-2\nu}\int d^\nu p[1-Y(p)]^{-1}\int d^\nu p\, Y(p)^2 .$$

The last integral is $(2\pi)^\nu/2\nu$ (since $\int \cos p_i \cos p_j = 0$). To bound the first integral we note that $1/(1-x)$ is convex and so, imitating the above analysis for $\nu \ge 3$,

$$(2\pi)^{-\nu}\int d^\nu p[1-Y(p)]^{-1} \le (2\pi)^{-3}\int d^3 p[1-\tfrac{1}{3}(\cos p_1+\cos p_2+\cos p_3)]^{-1} ,$$

which is finite. This proves our assertion.

The importance of the assertion that $\lim_{\nu\to\infty} I(\nu) = 0$ is that inequality (7) and the inequality $m^2 \le e_1$ (which follows from reflection positivity; see Ref. 4) then imply that as $\nu \to \infty$ the spontaneous magnetization approaches the energy e_1. As we will show below the energy e_1 converges to $\frac{1}{2}S^2$ as $\nu \to \infty$. Thus as $\nu \to \infty$ the spontaneous magnetization converges to the classical value. The same proof and conclusion applies to the Heisenberg antiferromagnet considered in Ref. 4. This result validates the folklore that as $\nu \to \infty$ the ground-state correlations converge to the classical values.

To show that $\lim_{\nu\to\infty} e_1 = \frac{1}{2}S^2$, we first note (as above) that $e_1 \ge \frac{1}{2}S^2$. To obtain an upper bound to e_1 we write the Hamiltonian (1) as the average of $\sum_{\langle xy\rangle}(-S_x^1S_y^1 - S_x^2S_y^2 + S_x^3S_y^3)$ and $\sum_{\langle xy\rangle}(-S_x^1S_y^1 - S_x^2S_y^2 - S_x^3S_y^3)$. The first Hamiltonian is unitarily equivalent to the antiferromagnetic Heisenberg Hamiltonian, and so Anderson's bound[1] says that its ground-state energy per bond is $\ge -(S^2+S/2\nu)$. The second Hamiltonian is the Heisenberg ferromagnet, and so its ground-state energy per bond is $-S^2$. Thus the ground-state energy per bond of Hamiltonian (1) is not less than $-(S^2+S/4\nu)$, and so $e_1 \le \frac{1}{2}S^2 + S/8\nu$. By combining the two bounds it follows that $\lim_{\nu\to\infty} e_1 = \frac{1}{2}S^2$.

One can also consider admixing some 3-component, i.e., the two-site interaction is changed to $-S_i^1S_j^1 - S_i^2S_j^2 + \Delta S_i^3S_j^3$. Then our method will show LRO for small Δ in all the above cases. It does not show LRO for

$\Delta=1$ with $\nu=2$, but it does for $\nu=3$, as shown in Ref. 4.

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[a]Permanent address: Mathematics Department, University of Arizona, Tucson, AZ 85721.

[b]On leave from Tata Institute of Fundamental Research, Bombay, India; now at AT&T Bell Laboratories, Murray Hill, NJ 07974.

[1]F. Dyson, E. H. Lieb, and B. Simon, J. Stat. Phys. **18**, 335 (1978).

[2]E. Jordao Neves and J. Fernando Perez, Phys. Lett. **114A**, 331 (1986).

[3]K. Kubo, Phys. Rev. Lett. **61**, 110 (1988).

[4]T. Kennedy, E. H. Lieb, and B. S. Shastry, J. Stat. Phys. (to be published).

[5]T. Matsubara and H. Matsuda, Prog. Theor. Phys. **16**, 569 (1956).

[6]E. Lieb and D. C. Mattis, J. Math. Phys. **3**, 749 (1962).