Chiral Perturbation Theory and Final-State Theorem

Tran N. Truong
Centre de Physique Theorique de l'Ecole Polytechnique, 91128 Palaiseau CEDEX, France
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Timelike scalar and vector form factors are recalculated using chiral perturbation theory and dispersion theory. It is shown that chiral perturbation theory at the one-loop level violates the final-state theorem (i.e., unitarity). In order to satisfy this theorem, chiral perturbation theory should be applied to the inverse of the form factor whose result is shown to be equivalent to the Padé-approximant method.

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There are two different approaches to implement the unitarity correction to the low-energy chiral theorems which are derived with use of the standard current-algebra technique or the low-energy effective Lagrangian.

The first one consists in the use of the well-known technique of the dispersion relation together with the unitarity to get some singular integral equations; in its simplest form, when the elastic unitarity is assumed, one gets the singular integral equation of the Muskhelishvili-Omnes (MO) type whose exact solution is known.[1] It corresponds to the summation of an *infinite* series of graphs with two particles in the intermediate state. The advantage of this nonperturbative method is that the final-state theorem on the phase of the calculated amplitude due to the strong interaction is exactly satisfied.[2]

This constraint is important, because it is model independent, and should be used to check the validity of the various approximations. This technique has been applied with much success in removing discrepancies between the current algebra or the low-energy effective chiral Lagrangian predictions and experimental results in $K\rightarrow\pi\pi ev$,[3] $\eta\rightarrow 3\pi$,[4] the $I=0$ S-wave pion-pion scattering length,[5] and the relation between $K\rightarrow 3\pi$, $K\rightarrow 2\pi$, and $K\rightarrow\pi$ amplitudes.[6] The discrepancies between theories and experiments are all of the same origin; namely, they are due to the neglect of the strong final-state interaction of the two pion or the multipion system.

The second method relies on the one-loop perturbation theory, with the strong-interaction pion-pion amplitude taken from an effective chiral Lagrangian, and is known as the chiral perturbation theory (CPT). Although it was first introduced about twenty years ago,[7] it only recently became popular thanks to an article by Weinberg[8] and more recently a rather complete analysis of Gasser and Leutwyler on the pion-pion scattering and the pion form factors.[9] Unlike the dispersion method, the CPT as implied by its name, is a perturbative method. Because of its complexity, practical calculations are limited to the one-loop approximation. The unitarity relation is not, in general, obeyed.

As a special example, it is shown in this Letter that (i) the usual one-loop CPT for the timelike form factor does not satisfy the unitarity or the final-state theorem.[10] This is a consequence of the perturbative approach and is equivalently a poor approximation to the solution of the MO integral equation. (ii) The CPT for the *inverse* of the form factor, on the other hand, is a much better approximation to the MO integral equation, and satisfies the final-state theorem to a very good accuracy. The calculated form factors with this method are very similar to the results given by the dispersion theory[3-6,10] and the Padé-approximation method discussed below.

Scalar pion form factor with use of dispersion theory.— Let us recall some well-known results of the strong-interaction physics in the late 1950's. With use of the analytic property of the form factor, it is straightforward to write a twice-subtracted dispersion relation for the scalar form factor $S(s)$,

$$S(s)=1+sS'(0)+\frac{s^2}{\pi}\int_{4\mu^2}^{\infty}\frac{S(s')e^{-i\delta_0}\sin\delta_0(s')ds'}{s'^2(s'-s-i\epsilon)}, \tag{1}$$

where μ is the pion mass, s is the momentum transfer, and the elastic unitarity is assumed; $\mathrm{Im}S(s)=S(s)\times\exp(-i\delta_0)\sin\delta_0$, δ_0 is the S-wave $I=0$ pion-pion phase shift. We normalize $S(0)=1$ for convenience, and by definition $S'(0)=\frac{1}{6}\langle r_s^2\rangle$, where r_s is the scalar rms radius of the pion. Equation (1) is a singular integral equation of the MO type. It will become obvious later why a twice-subtracted dispersion relation is used in Eq. (1); its formal solution is, however, independent of the number of the subtractions. The solution of the MO equation is well known, $S(s)=P_n(s)G(s)$, where

$$G(s)=\exp\frac{s}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta_0(s')\,ds'}{s'(s'-s-i\epsilon)}, \tag{2}$$

where $P_n(s)$ is an nth degree polynomial in s with $P_n(0)=1$. Because the chiral symmetry does not require that $S(s)$ vanish in the low-energy region, we assume that $P_n(s)=1$. The unitarity relation which states that, in the form of the final-state theorem, to first order in the weak or electromagnetic interaction, and to *all order* in the strong interaction, the phase of S is the strong-

interaction phase δ_0, is explicitly verified by Eq. (2). This enables us to calculate the scalar form factor S in terms of the experimental phase shift δ_0. [When the inelastic effect is included $G(s)$ can always be factorized.[11]] In practice, $G(s)$ is calculated by construction from the partial-wave dispersion relation.

Scalar pion form factor $S(s)$ with use of chiral perturbation theory.—Instead of solving exactly the integral equation (1), the CPT consists in setting $S(s)=1$ and $f_0^*(s)=\exp(-i\delta_0)\sin\delta_0/\rho(s)$ by the expression given by the Weinberg expansion,[12] $f_0(s)\cong f_0^W(s)=(L/2)(s-\mu^2/2)$; $\rho(s)$ is the phase-space factor and is equal to $(1-s/4\mu^2)^{1/2}$. With use of this approximation in Eq. (1), it follows that

$$S(s)=1+sS'(0)-(L/2)\{(s-\mu^2/2)[h(s)-h(0)]+(\mu^2/2)sh'(0)\}, \quad (3)$$

where $L=(4\pi f_\pi^2)^{-1}$, $f_\pi=133$ MeV, and for $s\geq 4\mu^2$, the function $h(s)$ is defined as

$$h(s)=\frac{2}{\pi}\left[\frac{(s-4\mu^2)}{s}\right]^{1/2}\ln\frac{\sqrt{s}+(s-4\mu^2)^{1/2}}{2\mu}-i\rho(s)$$

and $h'(s)$ is its first derivative. Equation (3) is the chiral perturbation result of Gasser and Leutwyler[9] which is obtained here in a straightforward manner by the first-order iteration of the integral equation (1) and without going through the renormalization procedure. This shows the advantage of the dispersion method and also the limitation of the one-loop CPT method.

With use of the "experimental" result for the scalar radius of the pion $\langle r_s^2\rangle=0.5$ fm^2, which is similar to the values obtained by Gasser and Leutwyler,[9] it is straightforward to calculate the phase of the scalar form factor. (We ignore here the uncertainties in determining $\langle r_s^2\rangle$.) To see the violation of the final-state theorem near the threshold in the CPT method, let us compute the corresponding scattering length a_0(ff) from the chiral perturbation result of $S(s)$, assuming that it has the correct phase as required by the final-state theorem. From Eq. (3), we have

$$a_0(\text{ff})=\tfrac{7}{4}\mu L[1+\tfrac{2}{3}\mu^2\langle r_s^2\rangle+11\mu^2L/3\pi]^{-1}. \quad (4)$$

From this equation one obtains $a_0=0.123\mu^{-1}$ which is much less than the scattering-length value of $0.20\mu^{-1}$ obtained by Gasser and Leutwyler from the corresponding scattering process. This result clearly indicates that the unitarity relation is not obeyed near the threshold.

At higher energy, the phase of S, as given by the CPT to one loop, differs significantly from the experimental phase shift δ_0 (Ref. 13), as shown in Fig. 1. There is, unfortunately, no direct experimental information on the modulus of the pion scalar form factor.

Vector pion form factor with use of chiral perturbation theory.—Proceeding similarly to the calculation of the scalar pion form factor, using a twice-subtracted

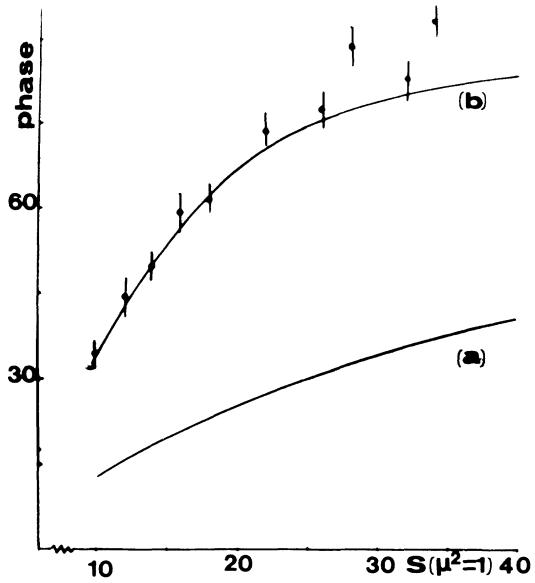

FIG. 1. Solid curves: Phases of the scalar form factors calculated by *a*, CPT Eq. (3) and *b*, improved CPT Eq. (9). Solid circles are some experimental S-wave, $I=0$, pion-pion phase shifts as given by Estabrooks *et al.* as quoted by Martin, Morgan, and Shaw in Ref. 13.

dispersion relation for $V(s)$ and using the elastic unitarity, we obtain the one-loop CPT result[9]:

$$V(s)=1+sV'(0)-(L/12)\{(s-4\mu^2)[h(s)-h(0)]+4\mu^2h'(0)s\}, \quad (5)$$

where $V'(0)=\frac{1}{6}\langle r_v^2\rangle$ and r_v is the vector rms radius of the pion. With use of the measured rms radius, $r_v^2=0.42$ fm^2, the phase and the modulus of the timelike pion form factor are calculated and shown in Fig. 2. It is seen that at a high value of s they differ significantly from the experimental data.

Higher loop effect.—Up to now, we have shown that the CPT series are inadequate at the one-loop level. This result is expected by examining the order of the perturbation for the ratio ImS/ReS. It is simple to see that, if one loop is included in ReS, to be consistent, the two loop must be included in the calculation of the ImS. More precisely, the perturbative unitarity relation for ImS including two loops is

$$\text{Im}S(s)=\rho(s)[S_1(s)f_1^*(s)+S_2(s)f_1^*(s)+S_1(s)f_2^*(s)], \quad (6)$$

where S_1 and S_2 are, respectively, the first and the remaining terms on the right-hand side of Eq. (3); they represent the tree and the one-loop amplitudes. Similarly, let f_1 and f_2 be the corresponding $I=0$ S-wave scattering amplitudes. The scattering length calculated with use of Eq. (6) in combination with the real part of

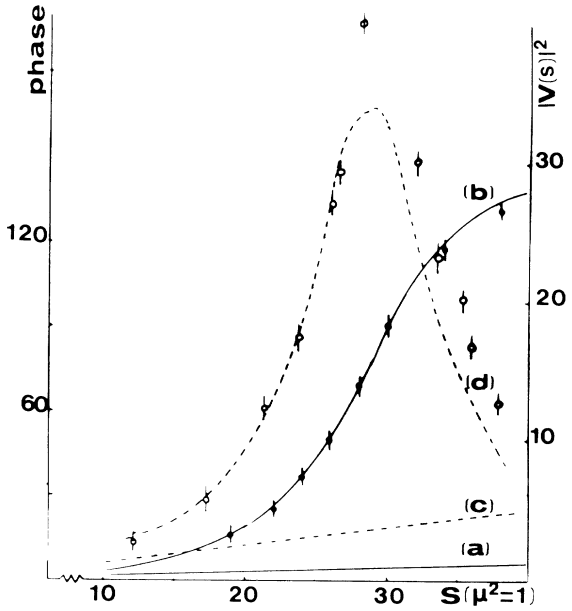

FIG. 2. Solid curves: Phases of $V(s)$ calculated by a, CPT Eq. (5) and b, improved CPT Eq. (10). Solid circles are some experimental P-wave pion phase shifts (Ref. 18). Dashed curves: $|V(s)|^2$ calculated by c, CPT Eq. (5) and d, improved CPT Eq. (10). Open circles are some experimental values of $|V(s)|^2$ (Ref. 18), (ρ-ω interference effect is not shown).

Eq. (3) is

$$a_0(\mathrm{ff}) = \tfrac{7}{4}\mu L\left\{1+\frac{\mathrm{Re}f_2(4\mu^2)}{f_1(4\mu^2)}\left[1+\tfrac{2}{3}\mu^2\langle r_s^2\rangle+\frac{11}{3\pi}\mu^2 L\right]^{-1}\right\}. \quad (7)$$

With use of the value $\mathrm{Re}f_2/f_1=0.25$ as given by Gasser and Leutwyler,[9] Eq. (7) yields $a_0(\mathrm{ff})=0.195\mu^{-1}$. This value is much closer to the value of the scattering length $0.20\mu^{-1}$ computed from the real part of the scattering amplitude, and hence the unitarity is satisfied in this case.

It can also be shown that if one calculated a_0 from the ratio of the imaginary to real part of the S-wave amplitude to one-loop order, one would arrive at a similar numerical result as Eq. (4) which is erroneous; the inclusion of the two-loop contribution to the imaginary part would remove this discrepancy.

From the above discussion, it is clear that one could try to include higher-loop effects to preserve perturbatively the unitarity condition. This method is, however, complicated and may not provide the proper answer to the problem because there is no guarantee that the ordinary perturbation series converge on the unitarity cut; for example, it is difficult to see how CPT even with many loops, can produce a resonance. In the following, we discuss a method to resum the perturbative series such that the *elastic* unitarity is obeyed and that the resonant effect can be taken into account.

Improved chiral perturbation.—We wish now to show that the chiral perturbation can be modified to satisfy the final-state theorem. The final results are similar to those given by Eq. (2). This can be done by writing down the dispersion relation for the *inverse* of the form factor. Let us consider, for example, the scalar form factor. (The vector form factor can be obtained in a similar manner.) After removing the zeros of the form factor, if they exist at all, we can write a twice-subtracted dispersion relation for the inverse of the form factor. With use of the fact that $\mathrm{Im}(S^{-1})=-\mathrm{Im}S/|S|^2$ and S is the solution of the MO equation as given by Eq. (2), it follows that

$$S^{-1}(s)=1-S'(0)s - \frac{s^2}{\pi}\int_{4\mu^2}^{\infty}\frac{e^{-i\delta_0}\sin\delta_0(s')S^{*-1}(s')\,ds'}{s'^2(s'-s-i\epsilon)}. \quad (8)$$

From Eq. (2), $S^*(s)$ has the phase $-\delta_0$ so that the phase factor is canceled out in the integrand. Similarly to the CPT for the form factor, the CPT for the inverse scalar form factor consists in setting $S=1$ and $\sin\delta_0/\rho=f_0^W(s)$ in the integrand of Eq. (8). Hence

$$S(s)=S_1(s)[1-S_2(s)S_1^{-1}(s)]^{-1}, \quad (9)$$

where S_1 and S_2 are, respectively, the first and the remaining terms on the right-hand side of Eq. (3). The scattering length calculated from this expression is $a_0=0.23\mu^{-1}$ and agrees well with the experimental results.[13,14] As can be seen from Fig. 1, the phase of $S(s)$ as given by Eq. (9) is also in a good agreement with the experimental data.

The reason that the chiral perturbation series is better with the inverse amplitude is due to the fact that $S^{*-1}f_0^*(s)\cong f_0^W(s)$ is a good approximation not only at threshold but also throughout the elastic region, even when the strong amplitude has a resonant character. This is so because, apart from some kinetical factor, $S(s)$ and $f_0(s)$ have the same phase and approximately the same energy dependence. To show this let us note that, from the general consideration of the analytic property of the elastic-scattering amplitude, one can write $f_0(s)=f_0^W(s)S(s)N(s)$, where f_0^W is the Weinberg tree amplitude and $S(s)=G(s)$ and is given by Eq. (2); $S(s)$ contains only the unitarity (right-hand) cut, and $N(s)$ contains the left-hand-cut singularity and is a slowly varying function in the region of the integration with $N(\mu^2/2)=1$. We can therefore set $N(s)\cong 1$. It follows that the numerator of the integrand of Eq. (8) is $f_0^W(s)\rho(s)$ as prescribed by the CPT for the inverse amplitude.

Equation (9) is a special case of calculating $G(s)$ with

the more general N/D method for the partial-wave dispersion relation when the left-hand cut is taken into account [i.e., the energy variation of $N(s)$]. This is why the CPT for the inverse of the form factor yields approximately the same results as the dispersion approach.[3-6,10]

The same modification also holds for the calculation of the vector form factor. In this case the improved CPT yields

$$V(s)=V_1(s)[1-V_2(s)V_1^{-1}(s)]^{-1}, \quad (10)$$

where V_1 and V_2 represent, respectively, the first and the remaining terms on the right-hand side of Eq. (5). The modulus and the phase of the form factor are calculated and plotted in Fig. 2. It is seen that the agreement between theory and experiment is excellent. In particular the ρ-meson mass is 760 MeV with a width of 142 MeV, which is in excellent agreement with the data. In fact, Eq. (10) is exactly the pion form factor given by Brown and Goble,[15] and Beg and Zepeda.[15]

The chiral perturbation theory as applied to the inverse of the amplitude is equivalent to summing an infinite series of the bubble graphs with the two-pion intermediate state. By this procedure, the unitarity is preserved and hence the final-state theorem is satisfied. The one-loop CPT calculations, Eqs. (3) and (5), are simply the first-order expansion of the right-hand side of Eqs. (9) and (10) and therefore inaccurate. It should be noticed that one cannot get a resonance behavior by expanding as an infinite series the denominator of S or V in these equations. This remark could be used as a suggestion of the nonconvergence of the CPT series on the unitarity cut.

Equations (9) and (10) are simply the diagonal [1,1] Padé approximants for the (renormalized) perturbation series.[16] We give here the *justification* for their validity. From the perturbation series for the partial-wave amplitude $f=f_1+f_2$, one can similarly construct the diagonal [1,1] Padé approximant: $f(s)=f_1(s)[1-f_2(s)f_1^{-1}(s)]$ which has the advantage over the ordinary perturbation series because the elastic unitarity relation $\mathrm{Im}f=\rho|f|^2$ is satisfied. The calculated value of a_0 from this equation, either by the real or the imaginary part method, yields exactly the same result, $a_0=0.21\mu^{-1}$ in a good agreement with Eq. (9). The Padé-approximant method should also be applied to the pion-pion scattering[9] at higher energy, the Ke_4, $\eta\to 3\pi$ (Ref. 17), and $K\to 3\pi$ problems in the CPT approach.

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