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Lyapunov Exponent for Quantum Dissipative Systems

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We define a Lyapunov exponent for a class of quantum dissipative systems which in the classical limit can undergo a cascade of period-doubling bifurcations into chaos. We do so by computing the average of a functional over a semiclassical trajectory for a dynamical system whose Poincaré section corresponds to the Hénon map. In the strongly dissipative limit we establish a scaling law which determines the way in which chaos can set in for finite values of Planck's constant.

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While it is known that interference effects suppress chaos in quantum conservative systems with time-dependent Hamiltonians,[1] the behavior of their dissipative counterparts can be quite different. Graham and Tél,[2] and Hida,[3] showed that for strongly dissipative systems, such as a classically kicked damped oscillator or a Josephson junction, quantum effects act as an external noise source on the underlying deterministic dynamics. Since there is a well established theory for the effect of fluctuations on the dynamics of simple non-area-preserving maps,[4] one can use those results to study the nature of quantum attractors.

Among the most useful notions of nonlinear dynamics is that of sensitive dependence on initial conditions. Briefly stated, when trajectories differing initially by a small amount diverge from each other in the course of time, the evolution of the system is said to be chaotic. Mathematically this is connected to the positivity of the so-called Lyapunov exponent, which measures the local tendency of nearby trajectories to either expand or contract onto each other. While this exponent is well defined for classical systems, it is not so in quantum mechanics, where the notion of trajectory loses its meaning. Nevertheless, since in the semiclassical limit the Wigner functions play the role of classical trajectories, one can attempt to define a Lyapunov exponent in such a limit and to study the onset of chaos as $\hbar$ becomes finite.[5]

In this paper we define a Lyapunov exponent for quantum dissipative systems which in the classical limit can undergo a cascade of period-doubling bifurcations into chaos. We do so by computing the average of a functional over a semiclassical trajectory for a dynamical system whose Poincaré section corresponds to the Hénon map. In our case the system is a pulsed harmonic damped oscillator. In the strongly dissipative limit we show that the aforementioned functional reduces to the one computed by Shraiman, Wayne, and Martin[6] for the case of logistic maps. Using the results of Ref. 4 for dynamical systems with noise, we establish a universal scaling law which determines the way in which chaos sets in when Planck's constant becomes nonzero. This implies that a system, which is classically integrable for a given control parameter, can be chaotic when its quantum aspects are considered, in contrast to well known results in conservative systems.

Consider a pulsed harmonic oscillator whose dissipation is determined by its coupling to a heat reservoir. The conservative part of the motion, in second quantized notation, is given by the Hamiltonian

$$H=\hbar\omega a^{\dagger}a-g\left[\left(\frac{\hbar}{2\omega}\right)^{1/2}(a+a^{\dagger})\right]_{n}\sum_{n=-\infty}^{\infty}\delta(t-n\tau)\,, \quad (1)$$

where a ($a^\dagger$) is the destruction (creation) operator for the oscillator, and $g(x)$ is the pulsing potential. Dissipation is assumed to be determined through the coupling of the oscillator to a heat reservoir through the interaction[7]

$$H_{\rm int}=\alpha\sum_i (aR_i^\dagger+a^\dagger R_i)\,, \tag{2}$$

where R_i ($R_i^\dagger$) are bath operators.[8] The distribution of the reservoir modes, which couple to the oscillator through the coupling constant α, is assumed to be such as to give the phenomenological damping constant γ. Since the system should be Markovian, this imposes the condition of weak coupling ($\gamma\ll\omega$) and high temperatures ($k_BT\gg\hbar\gamma$). With these assumptions, a master equation for the density matrix can be written, which leads to a Fokker-Planck-type equation for the Wigner function[9] of the damped oscillator in between pulses. Graham and Tél,[2] following an approach introduced by Berry *et al.*,[10] were able to construct a map which relates the Wigner function, W_{n+1}, after the $(n+1)$th kick to that after the nth pulse, W_n, by

$$W_{n+1}(x,y)=\int dx_n\int dy_n K(x,y;x_n,y_n)W_n(x_n,y_n)\,, \tag{3}$$

where $y=[-p+\omega f(x)/E]/\omega$, $f(x)=g'(x)$, $E=e^{-\gamma\tau}$, and K is a kernel given by

$$K(x,y;x_n,y_n)=\int\frac{d\xi}{2\pi}\int\frac{d\eta}{2\pi}\omega e^{-i\xi[x-x^0_{n+1}(x_n,y_n)]}e^{i\eta\omega[y-y^0_{n+1}(x_n,y_n;x)]}e^{-(\hbar Q/2\omega)(\xi^2+\omega^2\eta^2)}e^{iG(x,\eta;\hbar)}\,. \tag{4}$$

Here,

$$G=\hbar^{-1}[g(x+\tfrac{1}{2}\eta\hbar)-g(x-\tfrac{1}{2}\eta\hbar)-\eta\hbar g'(x)]\,, \tag{5}$$

with g defined in Eq. (1) and g' being its derivative,

$$Q=\frac{1-E^2}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right), \tag{6}$$

and

$$x^0_{n+1}(x_n,y_n)=-Ey_n+f(x_n)\,, \tag{7a}$$

$$y^0_{n+1}(x_n,y_n;x)=y^0_{n+1}(x_n,y_n)\,. \tag{7b}$$

The above equations were derived with the particular choice of τ such that $\omega\tau=2\pi(k+\frac{1}{4})$ with k an integer.

For suspensions of the Hénon map, $f(x_n)$ is given by any unimodal function with a quadratic maximum. Therefore, in the limit of strong dissipation ($\gamma\to\infty$) $E\to 0$, Eqs. (7) reduce to the standard logistic map.

Since quantum dissipation can be associated to classical maps with noise, we can use the field-theoretic method of Shraiman, Wayne, and Martin[6] to compute the Lyapunov exponent, λ. To do so, we need to calculate the correlation expression that defines it[11]:

$$e^{\lambda M}=\lim_{M\gg 1,\epsilon\to 0}\frac{x_M(x_0+\epsilon)-x_M(x_0)}{\epsilon}=f'(x_0)i\langle x_M\xi_0\rangle\,, \tag{8}$$

where

$$i\langle x_M\xi_0\rangle=Z^{-1}\int[Dx][D\xi]ix_M\xi_0\exp\left[\textstyle\sum_m\{i\xi_m[x_{m+1}-f(x_m)]-\frac{1}{2}\sigma^2\xi_m^2\}\right]. \tag{9}$$

Here $[Dx]$ is a discrete path integration over the sequence $x_{m+1}=f(x_m)+\xi_m$, and σ^2 is the variance of the Gaussian random variable ξ_m representing additive noise. In order to solve Eq. (9) for our quantum problem, we define the average of a functional $F[\{x,y\}]$ over a sequence $\{x,y\}$ by weighting each step with the kernel of the Wigner function given by Eq. (4). Thus,

$$\begin{aligned}\langle F[\{x,y\}]\rangle\equiv Z^{-1}&\int dx_N\int dy_N F(x_N,y_N)\int dx_{N-1}\int dy_{N-1}F(x_{N-1},y_{N-1})K(x_n,y_N;x_{N-1},y_{N-1})\\&\times\int dx_{N-2}\int dy_{N-2}F(x_{N-2},y_{N-2})K(x_{N-1},y_{N-1};x_{N-2},y_{N-2})\\&\times\cdots\times\int dx_1\int dy_1F(x_1,y_1)K(x_2,y_2;x_1,y_1)\int dx_0\int dy_0F(x_0,y_0)K(x_1,y_1;x_0,y_0)W_0(x_0,y_0)\,.\end{aligned} \tag{10}$$

Notice that rather than averaging over an arbitrary sequence we restricted the integral to points $\{x,y\}$ over the classical Hénon map, but weighted with K.

Since we are interested in the Lyapunov exponent associated with the x direction, we integrate the function F over the $\{y\}$ sequence and its conjugate $\{\eta\}$ to obtain

$$\begin{aligned}\langle F[\{x\}]\rangle=Z^{-1}&\int dx_N F(x_N)e^{iG(x_N,0;\hbar)}\int dx_{N-1}F(x_{N-1})\int\frac{d\xi_{N-1}}{2\pi}e^{i\xi_{N-1}[x_N-f(x_{N-1})]-\sigma^2\xi_{N-1}^2/2}e^{iG(x_{N-1},E\xi_{N-1}/\omega;\hbar)}\\&\times\prod_{n=1}^{N-2}\int dx_n F(x_n)e^{-E^2(i\xi_{n+1}x_n+\sigma^2\xi_{n+1}^2/2)}\int\frac{d\xi_n}{2\pi}e^{-i\xi_n[x_{n+1}-f(x_n)]-\sigma^2\xi_n^2/2}e^{iG(x_n,E\xi_n/\omega;\hbar)}\\&\times\int dx_0F(x_0)e^{-E^2(i\xi_1x_0+\sigma^2\xi_1^2/2)}\int\frac{d\xi_0}{2\pi}e^{-i\xi_0[x_1-f(x_0)]-\sigma^2\xi_0^2/2}\int dy_0W_0(x_0,y_0)\,.\end{aligned} \tag{11}$$

Taking the strong dissipation limit, $\gamma\tau \gg 1$, we can decouple the fluctuation variable ξ_n from ξ_{n+1}, reducing this expression to

$$\langle F[\{x\}]\rangle = Z^{-1}\int dx_N F(x_N)\, e^{iG(x_N,0;\hbar)} \prod_{n=1}^{N-1} \int dx_n F(x_n) \int \frac{d\xi_n}{2\pi} e^{-i\xi_n[x_{n+1}-f(x_n)]-\sigma^2\xi_n^2/2} e^{iG(x_n,0;\hbar)}$$
$$\times \int dx_0 F(x_0) \int \frac{d\xi_0}{2\pi} e^{-i\xi_0[x_1-f(x_0)]-\sigma^2\xi_0^2/2} \int dy_0\, W_0(x_0,y_0)\,. \quad (12)$$

Expanding this equation in terms of $\hbar^\alpha$, with $\alpha < 1$, and keeping the lowest terms makes the factors containing G drop out of it. Furthermore, in the limit $N \to \infty$ and $\int dy_0 W_0(x_0,y_0) = 1$, we obtain the general expression of Shraiman, Wayne, and Martin, which enables us to complete the calculation of Eq. (9) when $F[\{x\}]$ is substituted by $ix_m x_0$. From these results we see clearly that to lowest order in $\hbar$, and for strong dissipation, the quantum map becomes equivalent to a noisy one-dimensional recursion relation, in agreement with the predictions of Graham and Tél and of Hida. The noise terms are Gaussian, with zero mean and variance given by

$$\langle \xi_n \xi_m \rangle = \sigma^2 \delta_{n,m} = \frac{\hbar}{2\omega} \coth\left(\frac{\hbar\omega}{2k_B T}\right) \delta_{n,m}\,. \quad (13)$$

The above results allow us to use, in the semiclassical limit of strong dissipation, the scaling results[4] which relate the chaotic thresholds in the classical case to the noise strength. If $r^+(\sigma)$ and r_c denote the threshold values in the presence and absence of noise, respectively, then the Lyapunov exponent obeys the relation

$$\lambda(r_c - r^+;\sigma) = (r_c - r^+)^t \phi((r_c - r^+)^{-t}\sigma^\theta)\,, \quad (14)$$

with $t = 0.4498\ldots$ and $\theta = 0.37$. Since for a fixed level of noise $r_c - r^+ = \sigma\gamma$ with $\gamma = 0.75$, we see that in the semiclassical limit the chaotic threshold (signaled by a positive value of λ) is renormalized according to

$$r^+ = r_{\hbar=0} - \hbar^{0.37}\,, \quad (15)$$

with the Lyapunov exponent satisfying a scaling law in the vicinity of the transition, as $\hbar^{0.18}$.[12] Since it has been well established[4,6] that noise lowers the chaotic threshold, we infer that for a given control parameter, for which the system is classically integrable, the introduction of quantum effects will make it chaotic, contrary to well known results for Hamiltonian systems.[1]

We have thus succeeded at calculating the Lyapunov exponent for a class of quantum dissipative systems which in the classical limit have a period-doubling route to chaos. Notice that unlike the Hamiltonian case, where quantum interference effects suppress chaos,[1] dissipative systems can simulate a classical system with noise, thereby lowering the chaotic threshold (a positive Lyapunov exponent in a region where the classical system is integrable). This result seems to suggest that quantum effects make dissipative systems show a tendency to become chaotic at finite values of $\hbar$, as a result of the suppression of interference phenomena through coupling to a heat reservoir.

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[11]In this note the definition of the Lyapunov exponent through position correlations certainly depends on the reduction to a one-dimensional noisy map, where the noise is δ correlated. At this point, we believe that the definition is more general, and can be extended to a class of n-dimensional maps. This extension, which requires extensive analytic as well as numerical computations, is presently in progress.

[12]The cotangent does not diverge in the region of validity of the theory, i.e., $\hbar\omega$ is never much smaller than $k_B T$.