Monte Carlo Study of Fermion-Number Fractionization on a Lattice

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We study the fermionic charge fractionization in two dimensions on a lattice using the Monte Carlo method. In order to avoid the fermion-doubling problem we use the bosonized procedure, and the results show that the fractional charge remains when the quantum dynamics of the soliton is included. We find that the fractional charge shows up at different values of the coupling constant from those at which the symmetry breaking of the initial bosonic field appears, and then the theory looks as if there were two phase transitions.

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The fractionization of the fermionic number has played an important role in physics. Among its most important applications are the baryon number of the skyrmion, the catalysis action of the of 't Hooft–Polyakov monopoles, and the explanation of the anomalous states of polyacetylene found in the laboratories: The charged soliton has spin 0 while the neutral soliton exibits spin $\frac{1}{2}$.

From a theoretical point of view, the appearance of states with a fractional fermionic number[1] (that happens when, because of the fermion-soliton interactions, the resulting quantum states carry fermionic quantum numbers which can have fractional or even irrational values) is related to other effects such as the quantum Hall effect[2] and the existence of anomalies[3] (that is, the full set of the classical symmetries cannot be preserved in any of the many possible quantization schemes). All of these phenomena have a common origin: the existence of normalizable zero-energy modes of the Dirac elliptic operator $[i\partial - \Phi(x)]$. From a mostly physical point of view, these effects are due to the nontrivial and nonlocal deformations of the fermionic Dirac sea,[4] made by the boundary conditions that we must impose on the physical states of the domain of definition of the Hamiltonian in order to have a well defined Hermitian operator.[5]

In this work we resort to Monte Carlo simulations to study how the quantum fluctuations of the bosonic field affect the well known results obtained without including the quantum dynamics of the soliton, because, although there are topological reasons which could preserve the charge fractionization, it is not clear whether the effect could change or even disappear when the quantum fluctuations around the sòliton are taken into account.

We start with an Euclidian Lagrangian in 1+1 dimensions defined as

$$L_E = \int dx^2 [-\tfrac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi) + \tfrac{1}{2} m^2 \Phi^2 - \tfrac{1}{4}\lambda \Phi^4 + \bar{\psi}(i\partial\!\!\!/ + g\Phi)\psi] . \quad (1)$$

In order to avoid the problem of fermion doubling, which could mask the charge fractionization, we employ the bosonization procedure, which in 1+1 dimensions is implemented through the well known relations

$$i\bar{\psi}\partial\!\!\!/\psi = \tfrac{1}{2}(\partial_\mu \sigma)^2 , \quad (2a)$$

$$\bar{\psi}\gamma^\mu \psi = \pi^{-1/2} \epsilon^{\mu\nu} \delta_\nu \sigma , \quad (2b)$$

$$\bar{\psi}\psi = \zeta \cos(\beta\sigma) , \quad (2c)$$

$$i\bar{\psi}\gamma_5 \psi = \zeta \sin(\beta\sigma) , \quad (2d)$$

where $\beta = 2\pi^{1/2}$ and ζ represents a scale parameter.

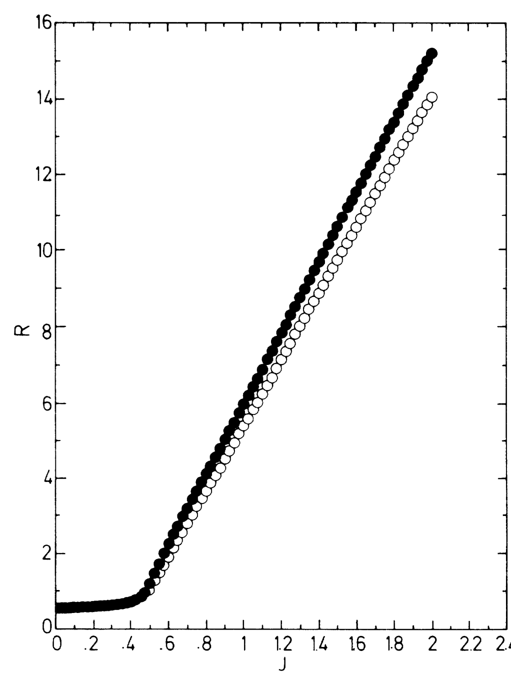

FIG. 1. R as a function of J when Φ is antiperiodic in the x direction and $G=1$. Open circles stand for a 8×8 lattice. Filled circles stand for a 16×16 lattice.

Thus in this representation the action is

$$S=J\sum_{n,\mu}\Phi(n)\Phi(n+\mu)-\Lambda\sum_{n}[\Phi^2(n)-1]^2-\sum_{n}[\Phi^2(n)+2\sigma^2(n)]+\sum_{n,\mu}\sigma(n)\sigma(n+\mu)-G\sum_{n}\Phi(n)\cos[\beta\sigma(n)]\,, \quad (3)$$

where we have rescaled the variable $\Phi(n)$ to $J^{1/2}\Phi(n)$ and we use the standard notation: μ is a unitary vector in the μ positive direction ($\mu=0,1$), and $n\in Z^2$ labels the site. The zero direction is the time.

The model exhibits a discrete global symmetry $\Phi\to-\Phi$, $\sigma\to\sigma+\frac{1}{2}\pi^{1/2}$ which can be spontaneously broken. Being interested in the phase diagram, and especially in the emergence of quantum fractional charge, we compute the observables

$$P=(1/V)\sum_n\langle\Phi(n)\rangle\,, \quad (4a)$$

$$R=(1/V)\sum_n\langle\Phi^2(n)\rangle\,, \quad (4b)$$

$$S=(1/V)\sum_n\langle\sigma^2(n)\rangle\,, \quad (4c)$$

$$Q=(1/T)\sum_n\langle\psi^\dagger(n)\psi(n)\rangle$$
$$=\pi^{-1/2}[\langle\sigma(K)-\sigma(-K)\rangle]\,, \quad (5a)$$

$$Q_S=(1/V)\sum_n\langle\bar\psi(n)\psi(n)\rangle$$
$$=(1/V)\sum_n\langle\cos[\beta\sigma(n)]\rangle\,, \quad (5b)$$

where T represents the size of the lattice in the time direction and $V=TL$ represents the volume; moreover we take $K=(L/2)-1$ in order to minimize the boundary effect.

The Monte Carlo simulation is made by use of the Metropolis algorithm in 8×8, 16×16, and 32×32 lattices. For the bosonic fields we use 2000 discretized values and we have controlled that $R^{1/2}$ and $S^{1/2}$ are always much smaller than the maximum of our discretized field.

For the σ and Φ fields, we have imposed periodic boundary conditions in the time direction, and we work with either periodic or antiperiodic boundary conditions in the spatial direction. We have checked that the numerical results for Q are independent of the spatial boundary conditions used for the field σ when we work on a 16^2 lattice. When the ϕ field is antiperiodic in the spatial direction and we are in the broken phase, the typical configuration is a kink. In this case, the center of the kink can travel through the lattice, and because the charge is localized in the region around the center of the kink [where the $\langle\sigma(n)\rangle$ changes], in order to eliminate asymmetry effects, it is convenient to center the kink. To do that, every time we want to apply the measure procedure, using the fact that the theory is translationally invariant, we move the entire Φ configuration in the x direction through the lattice, fixing the origin with the criterion that P be minimum, and then we thermalize the σ field again.

For the constant-coupling values we fix Λ and G ($\Lambda=0.1$, $G=1$, 0.25, 0.1), and J varies between 0.1 and 2. Once the system is thermalized, 5000 iterations are done, measuring every 20 sweeps.

The exact treatment of the problem makes possible a good control of errors, and the results of the 8^2 versus 16^2 lattice show that the finite-size effects are small. In order to control our numerical results, we have checked them with those of a semiclassical strong- and weak-coupling expansion in the regions of small and large J, respectively. The naive classical limit of this model has a potential $V(\Phi)$ with only one minimum for $J<\frac{1}{2}-\Lambda$ and a potential with two minima for $J>\frac{1}{2}-\Lambda$. In the $J\to0$ region, we can expand the kinetic term in the exponential of S (3), and to second order in J we obtain

$$R=[\rho^2]+J^2[\rho^2][\rho^4]+O(G)\,, \quad (6)$$

where

$$[\rho^n]=\frac{\int_{-\infty}^{+\infty}\rho^n e^{-\Lambda(\rho^2-1)^2-\rho^2}d\rho}{\int_{-\infty}^{+\infty}e^{-\Lambda(\rho^2-1)^2-\rho^2}d\rho}\,, \quad (7)$$

which can be evaluated analytically as a quotient of parabolic cylindric functions. For $\Lambda=0.1$, we obtain

$$R=0.475+0.602J^2+O(G)\,, \quad (8)$$

in good agreement with the Monte Carlo results for $J\ll\frac{1}{2}-\Lambda$.

In the large-J region, the dominant configuration for periodic boundary conditions in the x direction for the fields Φ and σ is the solution of the equations

$$2\Lambda\Phi_0^3-(2J+2\Lambda-1)\Phi_0-(G/2)\cos(\beta\sigma_0)=0\,, \quad (9a)$$

$$-G\beta\Phi_0\sin(\beta\sigma_0)=0\,, \quad (9b)$$

which gives

$$2\Lambda\Phi_0^3-(2J+2\Lambda-1)\Phi_0\pm G/2=0\,, \quad (10a)$$

$$\sigma_0=(n/2)\pi^{1/2}\,, \quad (10b)$$

where the sign + or − in (10a) corresponds to even or odd in (10b). When the boundary conditions for the field Φ are antiperiodic (in the x direction) the value of Φ_0 is modified, and we have

$$R=\Phi_P^2\left\{1-\tanh\left[\frac{L}{2}\left(2+\frac{2\Lambda}{J}-\frac{1}{J}\right)^{1/2}\right]\Big/\frac{L}{2}\left(2+\frac{2\Lambda}{J}-\frac{1}{J}\right)^{1/2}\right\}, \quad (11)$$

with L the spatial size of the lattice and Φ_P the solution of (10a) in the periodic case. The agreement between R from (8) for the periodic case and from (11) for the antiperiodic case with the Monte Carlo results is excellent.

When the boundary conditions for $\Phi(n)$ are periodic, the system has two phases and then P, as defined in (4a), is a good order parameter. In this case, the fermionic charge Q is zero in both phases, and Q_S is zero only in the unbroken phase.

In the case when we take antiperiodic boundary conditions for the Φ field, Q_S and P are zero in both phases (since in the phase where the symmetry is broken we have centered the kink before measuring). The values for R (4b) in our Monte Carlo simulations ($G=1$) are shown in Fig. 1; there, one can see that the corrections due to the lattice size are not important when we go from a 8×8 to a 16×16 lattice and further, the existence of a phase transition at values of J around 0.5; the nature of this phase is second order. It should be noted that in comparing the values of R obtained in this version with those obtained using a discretized version of (1) with Kogut-Susskind fermions,[6] one can see that they are nearly equal for $\zeta=\pi^{-1}$, which gives the mass scale in accordance to semiclassical results.[7] Figure 2 shows R for different values of G and J varying between 0.1 and 2; there one sees that when G goes from 1 to 0.1, the critical value of J (at which the breaking of the symmetry $\Phi\to -\Phi$ happens) changes slowly from 0.5 to 0.625.

The values of the fermionic charge Q for antiperiodic boundary conditions in the x direction of the field Φ are shown in Fig. 3 for a 16×16 lattice, and for $G=1$ and 0.25; we have suppressed the error bars to points since they are always smaller than the dot limits. There it is shown that for values of $G=1$ and J less than 0.5, the fermionic charge Q is zero; this is the expected result since for those values of J the symmetry $\Phi\to -\Phi$ is not broken. In this region the net charge of each of the dominant configurations is small, and as a consequence, the statistical errors are small. When $G=1$ and $J>0.5$, the symmetry $\Phi\to -\Phi$ is broken and the numerical results show that Q takes values around $\pm\frac{1}{2}$ with very small statistical errors; the sign of Q depends on the thermalization process but it is stable when the system is already thermalized. Then we conclude that for those values of G and J, the fermionic charge keeps its "semiclassical" fractional value $\frac{1}{2}$ once the dynamics of the soliton is included.

The numerical results for $G=0.25$ and $J<0.6$ or $J>1$ are similar to those described previously, but when $0.6<J$ and $J>1$ the result $Q=0$ is surprising since for these values the symmetry $\Phi\to -\Phi$ is broken. We have checked that the dominant configurations are kinklike (or antikink) with an extended and irregular center. At first we could think that since we are working on a finite lattice, a kink can pass through the border and become

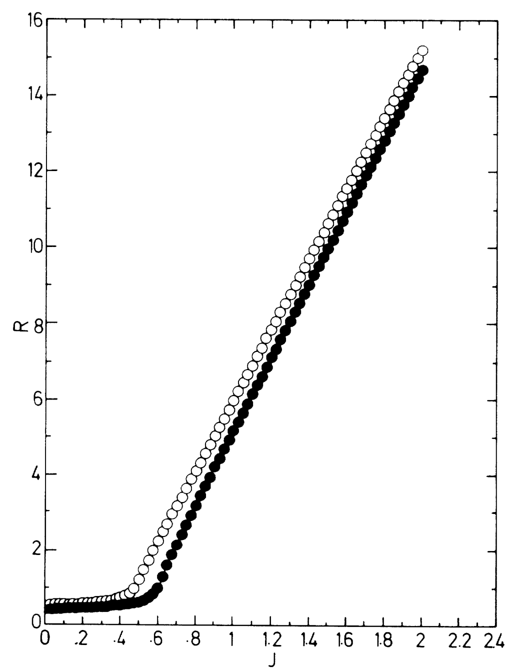

FIG. 2. R as a function of J when Φ is antiperiodic in the x direction. Open circles stand for $G=1$. Filled circles stand for $G=0.1$.

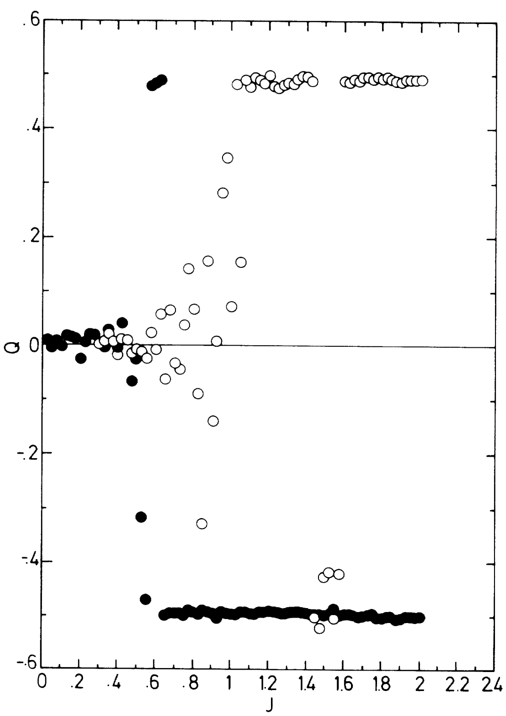

FIG. 3. Fermionic charge on a 16×16 lattice (antiperiodic Φ). Open circles stand for $G=0.25$. Filled circles stand for $G=1$.

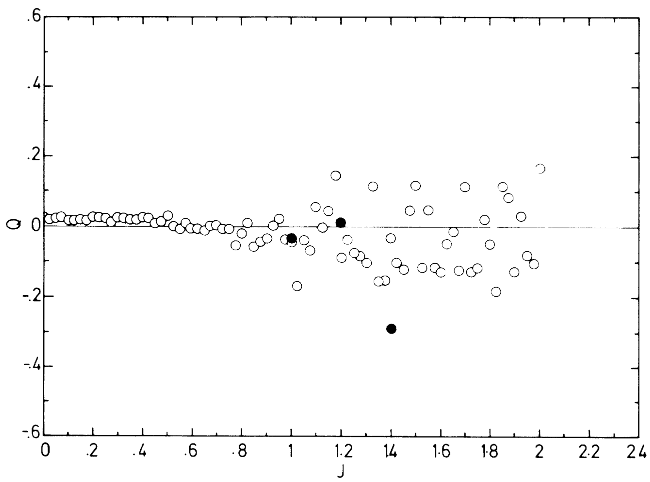

FIG. 4. Fermionic charge on a 16×16 lattice (open circles) and on a 32×32 lattice (filled circles) (antiperiodic Φ). $G=0.1$.

an antikink, which would give rise to cancellations and could explain the result; however, this is not possible on our simulation since we are always centering the configuration before measuring. Keeping in mind that we are on a 16×16 lattice, we conclude that there is a range of J values where the system has a broken symmetry with $Q=0$, which means that the theory has two phase transitions. This is supported by the values of Q shown in Fig. 4; in the region where $G=0.1$ and $0.625<J<2$, the symmetry $\Phi\rightarrow-\Phi$ is broken and the fermionic charge is always compatible with zero, for both the 16^2 and 32^2 lattice. In this situation we have seen that with $1<J<1.8$ the Φ field presents the typical kink profile with a width extended over one or two lattice kinks. However, the σ field does not exhibit the kink form since the height of the associated potential (approximately $G\langle\Phi\rangle$) takes a small value and then the quantum fluctuations produce an important tunneling effect which annihilates the hypothetical fractionization phenomenon.

On the other hand, the effect due to the lattice finite size is not so important. In fact, when the Φ field exhibits a sharp kink profile we can make a semiclassical estimation of the width associated with the hypothetical σ kink; for $x>0$ this solution would be

$$\sigma(x)=\arctan\{\exp[(g|\Phi|)^{1/2}x]\}. \tag{12}$$

Looking for a fermionic charge measure with 10% of error we obtain the following condition:

$$L_W>\frac{3.7}{(g|\Phi|)^{1/2}}, \tag{13}$$

which provides us a value for L_W in the 6–10 range. Then the effect associated to the finite size is not a fundamental factor to detect the fractionization phenomenon in our lattice. The study of the finite-size effects as well as a better characterization of the phases of the theory, especially around the critical points, which is the relevant region for the continuum limit, is in progress.

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