Comment on "Exact Critical Behavior of a Random-Bond Two-Dimensional Ising Model"

In a recent Letter[1] Shankar has computed the ensemble average of the square of the spin-correlation function for the critical 2D Ising model with quenched disorder using bosonization. In the penultimate paragraph, Shankar points out that he finds self-averaging of all *even* moments to leading order as $R \rightarrow \infty$, i.e., for $N=$even

$$\langle\langle s(\mathbf{0})s(\mathbf{R})\rangle^N\rangle_{\mathrm{Av}} \sim \langle\langle s(\mathbf{0})s(\mathbf{R})\rangle^2\rangle_{\mathrm{Av}}^{N/2} \sim R^{-N/4}(\ln R)^{N/8}, \qquad (1)$$

suggesting sample independence of the correlation function asymptotically. On the basis of this result, he argues that the first moment be given by (1) with $N=1$.

I point out in this Comment that I have computed[2] *all* moments of the random spin-correlation function at the transition to leading order using a different method, developed in Ref. 3. I obtain

$$\langle\langle s(\mathbf{0})s(\mathbf{R})\rangle^N\rangle_{\mathrm{Av}} \sim R^{-N/4}(\ln R)^{N(N-1)/8}, \qquad (2)$$

$$RN=1,2,3,\ldots,$$

which leads to a physical picture very different from Shankar's (see below).

My calculation is exact in the same sense as Shankar's: in both the leading behavior as $R \rightarrow \infty$ is determined exactly by a one-loop renormalization-group calculation. Both disagree with Ref. 4.

My result satisfies the necessary inequalities, e.g.,

$$\langle\langle s(\mathbf{0})s(\mathbf{R})\rangle^{2N}\rangle_{\mathrm{Av}} \geq \langle\langle s(\mathbf{0})s(\mathbf{R})\rangle^N\rangle_{\mathrm{Av}}^2. \qquad (3)$$

I predict pure behavior for the first moment [$N=1$ in Eq. (2)] in disagreement with Shankar's prediction. My result also disagrees with Shankar's predictions for the higher moments in that it exhibits *non-self-averaging.* I have noticed[5] that the origin of this discrepancy lies in an error in Shankar's calculation of the moments with even $N>2$ in which he has omitted the contribution of the term

$$(g^2/2)\sum_{i\neq j}^{N/2}\cos(2\pi^{1/2}\phi_i)\cos(2\pi^{1/2}\phi_j)$$

[Eq. (21) of Ref. 1]. I have recovered[5] my result for the even moments using the bosonization approach as well. For $N=2$, Eqs. (1) and (2) agree since this term does not exist.

Physical interpretation: Since the correlation function $G=\langle s(\mathbf{r})s(\mathbf{r}+\mathbf{R})\rangle_{J_{ij}}$ is not extensive as $R \rightarrow \infty$, sample-to-sample fluctuations are not expected to vanish as $R \rightarrow \infty$ [in fact,

$$(\langle G^2\rangle_{\mathrm{Av}} - \langle G\rangle_{\mathrm{Av}}^2)/\langle G\rangle_{\mathrm{Av}}^2 \sim (\ln R)^{1/4} \rightarrow \infty,$$

Eq. (3)]. In order to obtain the behavior of G as $R \rightarrow \infty$ for a typical sample of bonds, one should average $\ln G$ which I sketch now (see Ref. 2): Define the probability distribution

$$P_R(G) = \mathcal{N}^{-1}\langle\delta(G - \langle s(\mathbf{r})s(\mathbf{r}+\mathbf{R})\rangle_{J_{ij}})\rangle_{\mathrm{Av}}, \qquad (4)$$

which samples the correlation at fixed R. The Nth moment [Eq. (3)] is then given by $\langle G^N\rangle_{\mathrm{Av}}(R) = \int G^N P_R(G)dG$, also for $N\neq$integer, and varies smoothly with N. One can therefore expect the continuation of Eq. (3) to $N=0$ *to be safe.* Differentiation at $N=0$ shows that $\ln G$ has a normal distribution with mean $-\frac{1}{4}\ln R - \frac{1}{8}\ln\ln R$ and variance $\frac{1}{4}\ln\ln R$. Hence G decays for a typical sample like

$$G_{\mathrm{typ}}(R) = \langle s(\mathbf{r})s(\mathbf{r}+\mathbf{R})\rangle_{J_{ij}} \underset{R\rightarrow\infty}{\longrightarrow} e^{\langle \ln G\rangle_{\mathrm{Av}}} \sim R^{-1/4}(\ln R)^{-1/8}. \qquad (5)$$

The fluctuations of $\ln G$ about its average vanish as $\sim 1/(\ln\ln R)^{1/2}$ [more precisely,

$$P_R(G)dG = \tilde{P}_R(\alpha)d\alpha = \mathcal{N}^{-1}\exp\{-f(\alpha)\ln\ln R\},$$

where $\alpha = -[\ln(GR^{1/4})]/\ln\ln R$, $f(\alpha) = 2(\alpha - \frac{1}{8})^2$ is universal, $\mathcal{N}=$normalization].

To summarize, I have computed the statistics of the spin-correlation function G at the transition as the sample of bonds $\{J_{ij}\}$ fluctuates. It reflects the nonextensive nature of G. There are two physically relevant correlation functions in the random system: (i) For a typical fixed sample, G behaves like $G_{\mathrm{typ}}(R)$ as of Eq. (5) as $R \rightarrow \infty$. (ii) The averaged correlation function, relevant for the susceptibility, is asymptotically equal to the pure one:

$$[G]_{\mathrm{Av}}(R) \sim G_{\mathrm{pure}}(R) \sim R^{-1/4} > G_{\mathrm{typ}}(R).$$

The $\ln\ln\tau$ behavior of the specific heat was also recovered in Ref. 3.

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