Simple Model of Collective Transport with Phase Slippage

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We present a mean-field analysis of a many-body dynamical system which models charge-density-wave transport in the presence of random pinning impurities. Phase slip between charge-density-wave domains is modeled by a coupling term that is periodic in the phase differences. When driven by an external field, the system exhibits a first-order depinning transition, hysteresis, and switching between pinned and sliding states, and a delayed onset of sliding near threshold.

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Collective transport in coupled dynamical systems is a topic of considerable current interest.[1] An experimental example is the nonlinear conduction seen in charge-density-wave (CDW) samples.[2,3] When a sufficiently strong dc electric field is applied to a sample with a static CDW, the CDW depins from impurities in the lattice and begins to slide and carry current. Classical models of CDW transport[4-10] assume that the dynamics are dominated by competition between the internal elasticity of the CDW and the local potentials of randomly spaced impurities.

These models of CDW transport do not account for experimentally observed hysteresis, switching, and delayed conduction in "switching samples."[11-15] CDW inertia,[16] current noise,[17] and avalanche depinning[18] have been proposed to account for switching. More recently, switching and hysteresis have been ascribed to phase slippage in the CDW.[12-15] A physical model for a CDW in a switching sample is a collection of domains, each with a well-defined phase, separated by regions where the amplitude of the CDW is weak.[12,15] Phase slip can occur easily in these connecting regions. A rigorous theory of CDW transport for this case is very difficult, although a detailed analysis of a phenomenological model with a few degrees of freedom has been presented.[15] It is also uncertain which of the observed complex phenomena are intrinsic and which are properties of particular samples or experimental treatments.

In this Letter we present a simple model of collective transport which is applicable to CDW transport in switching samples. The model consists of many phases which represent the states of CDW domains, and phase slip due to amplitude collapse[15] is modeled by a weak-coupling term periodic in the phase differences. This is a simple modification of a well-understood model[6-8] with elastic coupling and no phase slip. As we will show, the periodic coupling gives rise to switching, hysteresis, and delayed conduction. Our approach is to analyze a simple model which may have some generality, rather than to make a detailed phenomenological treatment specific to charge-density waves.

The Hamiltonian is

$$H=\frac{J}{2N}\sum_{i,j}[1-\cos(\theta_i-\theta_j)]+b\sum_j[1-\cos(\theta_j-\alpha_j)]\,, \tag{1a}$$

and we assume zero temperature and relaxational dynamics with a driving field

$$\dot{\theta}_j=-\frac{\partial H}{\partial\theta_j}+E_0,\quad j=1,\ldots,N\,. \tag{1b}$$

The θ_j represent the phases of weakly coupled domains.[8,9] In other models,[6,7] θ_j represent the phase distortion of the CDW at the jth pinning site. In Eq. (1), J is the coupling strength, b is the pinning strength, α_j is a pinning phase randomly distributed on $[-\pi,\pi]$, and E_0 is an electric field applied along the CDW wave vector. The coupling term favors phase coherence, whereas the random fields try to pin each θ_j at α_j. For weakly coupled domains, the ratio $K=J/b$ is small. The infinite-range coupling in Eq. (1) corresponds to the mean-field approximation, also used for previous work.[6-8]

The model (1) is closely related to the system studied by Fisher.[6] The difference is that in the Hamiltonian Eq. (1a) we have assumed a periodic coupling $1-\cos(\theta_i-\theta_j)$ rather than a quadratic coupling $\frac{1}{2}(\theta_i-\theta_j)^2$. The periodic coupling in Eq. (1a) allows for phase-slip processes[6,11] and corresponds physically to the effects of CDW defects[1,19] or amplitude collapse[12,15] between coherent regions of the CDW. In particular, the model is appropriate in CDW systems with strong pinning centers that favor the formation of weakly coupled domains.[12,20] We have made the simplifying assumption that the argument of the periodic coupling is the phase difference $\theta_i - \theta_j$ rather than a more general multiple $\lambda(\theta_i - \theta_j)$, where λ reflects the amount of polarization that can be built up before phase slip occurs. Additional metastable states[10] with different polarizations can exist below the depinning threshold for $\lambda \neq 1$; these are not present in our model.

We first consider the static configuration of the system when $E_0 = 0$. The phase coherence of the equilibrium configurations depends on the normalized coupling strength K. For instance, in the absence of coupling ($K=0$), the θ_j are pinned at α_j and are completely incoherent, whereas for $K \to \infty$, there is perfect coherence ($\theta_i = \theta_j$ for all i,j). To characterize the transition from incoherence to coherence, we define a complex order parameter

$$re^{i\phi} = N^{-1} \sum_j \exp(i\theta_j),$$

where r measures the coherence and ϕ is the average phase.

We now show analytically that there is a first-order transition in the model from the incoherent state ($r=0$) to the coherent state ($r \approx 1$) at $K=2$, when the domains are strongly coupled. This zero-field transition is an artifact of mean-field theory, but a related hysteretic transition occurs for nonzero E_0 in the physically relevant weak-coupling regime, as discussed below. The strategy of the analysis is to derive a self-consistent equation for r, by use of the fact that r determines the equilibrium phases θ_j and is in turn determined by them.

Equilibria of H satisfy $\partial H / \partial \theta_j = 0$, i.e.,

$$\sin(\alpha_j - \theta_j) + \frac{K}{N} \sum_i \sin(\theta_i - \theta_j) = 0.$$

Rewriting the sum in terms of the order parameter yields

$$\sin(\alpha_j - \theta_j) + Kr \sin(\phi - \theta_j) = 0. \tag{2}$$

We may set $\phi = 0$ because there is no absolute phase reference. This choice removes the rotational degeneracy. Solving Eq. (2) for θ_j yields

$$e^{i\theta_j} = \left(\frac{Kr + e^{i\alpha_j}}{Kr + e^{-i\alpha_j}} \right)^{1/2}. \tag{3}$$

Combining Eq. (3) with $r = N^{-1} \sum_j \exp(i\theta_j)$ and letting $N \to \infty$, we obtain the self-consistency relation for r:

$$r = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Kr + \cos\alpha}{(1 + 2Kr\cos\alpha + K^2 r^2)^{1/2}} d\alpha. \tag{4}$$

For each K, the values of r that satisfy Eq. (4) may be found as follows [Fig. 1(a)]. Let $u = Kr$ and let $f(u)$ denote the integral in Eq. (4), which may be expressed exactly as an elliptic integral.[21] Because $f(u)$ and u/K are both equal to r, the intersections of $f(u)$ and the line u/K yield solutions for the coherence r, given the normalized coupling strength K [Fig. 1(a)].

Figure 1(b) shows the first-order transition between incoherent and coherent states. The incoherent state

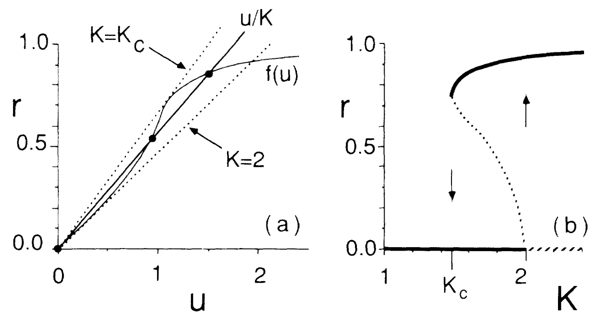

FIG. 1. (a) Solid lines indicate the integral $f(u)$ plotted from Eq. (4) together with the line u/K (see text). Equilibrium solutions for r occur where $f(u)$ intersects the line u/K. For the value of K shown, three solutions exist (filled circles). Dashed lines show u/K for the critical values $K = K_c$ and $K=2$. (b) Plot of the exact equilibrium solutions for r vs K: solid lines, locally stable equilibria; broken lines, unstable equilibria.

$r=0$ always solves Eq (4). An unstable second branch of the solutions bifurcates from $r=0$ at $K=2$, with $r \sim (2-K)^{1/2}$ as can be seen from Eq. (4) and the series expansion $f(u) = u/2 + u^3/16 + O(u^5)$, valid for $u < 1$. We have also proven[21] that $r=0$ is locally stable for $K<2$ and unstable for $K>2$. A locally stable third branch of solutions, with $r \approx 1$, is created when u/K intersects $f(u)$ tangentially [Fig. 1(a)] at $K = K_c \approx 1.489$. Note that for K between K_c and 2, the system is bistable. We emphasize that this first-order transition is a consequence of the cosine coupling in Eq. (1a) and would not be seen if a quadratic coupling were assumed.

We turn now from statics to dynamics. In the presence of a driving field, the equations of motion from Eq. (1b) are

$$d\theta_j/dt = E + Kr\sin(\phi - \theta_j) + \sin(\alpha_j - \theta_j). \tag{5}$$

By letting $E = E_0/b$ and time $t \to bt$, we have set $b=1$ without loss of generality; as before, $K = J/b$. The second term on the right-hand side of Eq. (5) is the collective torque exerted on θ_j by all other phases. For $E=0$ and small K, the phase coherence $r=0$ and therefore the collective torque is zero. If r becomes nonzero, the collective torque provides a positive feedback which tries to increase r further by aligning each θ_j with the average phase ϕ. The physical consequences of this process are hysteresis and delayed conduction, as discussed below. In our model hysteresis and switching result from the transition to coherence of a randomly pinned state. Incoherence of the pinned state occurs naturally for a large number of random pinning phases α_j; systems as small as three phases show hysteresis and switching, but only when the α_j are chosen evenly spaced on $[-\pi, \pi]$. Thus in our model these phenomena are associated with many degrees of freedom.

Figure 2 plots the regions of stability of the pinned and sliding CDW states. The pinned state ($d\theta_j/dt = 0$) in this model can be analytically shown to be incoherent ($r=0$). Using variational stability analysis about the pinned state, we have proven[21] that this state becomes unstable above the depinning threshold field $E_T = (1 - K^2/4)^{1/2}$ when $K < 2$, as shown in Fig. 2. For strong coupling, $K > 2$ where the model is not physically relevant, $E_T = 0$ and the CDW slides ($d\phi/dt > 0$) for any fields $E > 0$. This is an artifact of mean-field theory which also occurs in models[6-8] with elastic coupling. Numerical solutions of Eq. (5) show that the sliding state is always coherent ($r > 0$). The sliding state becomes pinned and incoherent below a separate pinning threshold E_P shown as the dashed line in Fig. 2, which was calculated numerically from the initial condition $r = 1$. This boundary extends from the critical value K_c found analytically for $E = 0$, also shown in Fig. 1. The solid and dashed lines in Fig. 2 bound a hysteretic region where both pinned and sliding solutions are stable; the final state reached depends on the initial conditions. The physically relevant weak-coupling region of Fig. 2 is for $K < K_c$, where E_T and E_P are nonzero.

The model predicts hysteresis and switching between pinned and sliding states as illustrated by the numerical solutions of Eq. (5) shown in Fig. 3. As E is increased slowly past E_T, the induced collective velocity $d\phi/dt$ corresponding to the CDW current jumps up discontinuously, then increases nearly linearly. If E is then decreased, the velocity $d\phi/dt$ decreases and then drops discontinuously to zero at the separate pinning threshold $E = E_P$ as shown in Fig. 3. When the CDW pins, the coherence r also drops discontinuously to zero. This loss of coherence is illustrated in the limit $E_P = 0$ for the analytical results in Fig. 1(b). Hysteretic current-voltage curves have been seen in low-temperature experiments on CDW samples with dilute impurities or irradiation-induced defects, which act as dilute, strong pinning sites.[11-14] The switching and hysteresis predicted by the model depend crucially on the periodic coupling in Eq. (1a); neither switching nor hysteresis are predicted for quadratic coupling.[6-8]

The model exhibits delayed conduction above the depinning threshold E_T when $E < 1$. Numerical solutions of Eq. (5) were used to study the evolution of the system from a random initial state. The system first rapidly reaches an incoherent configuration with $\theta_j \approx \alpha_j + \sin^{-1}E$, then gradually develops coherence, and finally depins suddenly when r becomes appreciable.[21] The delay before depinning increases near the threshold E_T, as observed experimentally.[11,14] If $E > 1$, switching occurs immediately.

Numerical solutions of Eq. (5) show that the individual phases do not move with a constant velocity in the sliding state, although the collective velocity $d\phi/dt$ is constant. Near threshold, the motion of each phase is periodic, alternating between rapid advances by nearly 2π, and slow creep about its pinning phase. In this respect, Eq. (5) and other mean-field models[6-8] agree with results from more realistic short-range models,[10] and with recent experiments[12,20] which suggest a spatially nonuniform rate of CDW phase advance near threshold. An aritfact of the mean-field approximation is that all the phases θ_j execute identical periodic motions shifted in time and phase.

We have also analyzed the dynamics of Eq. (5) far above the depinning threshold. For $E \gg E_t$, perturbation theory[21] yields $(d\phi/dt)/E = 1 - (1/2E^2) + O(E^{-4})$. Thus, the deviation from the limiting dc conductivity as $E \to \infty$ is proportional to E^{-n} with $n = 2$, in agreement with some CDW models[6,8,22] and in contrast to the value $n = \frac{1}{2}$ predicted by others.[5] The available data for high-field conductivity in CDW's[23] suggest $n \approx 1-2$.

Simplification of approximations in the model are infinite sample size N and infinite-range coupling. Solu-

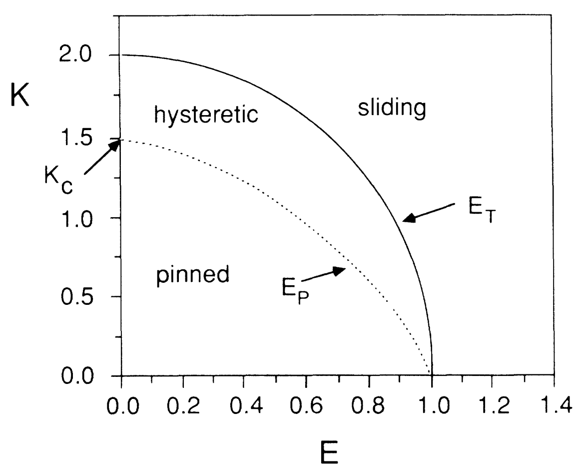

FIG. 2. Stability diagram for the model Eq. (5): solid line, depinning threshold $E_T = (1 - K^2/4)^{1/2}$ determined analytically; dashed line, pinning threshold E_P obtained by numerical integration of Eq. (5). Note the presence of hysteretic region.

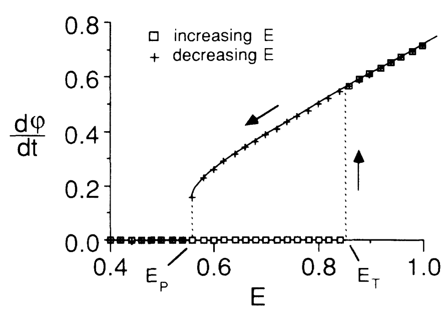

FIG. 3. Hysteresis and switching between pinned and sliding states. Data points obtained for $N = 300$ phases by numerical integration of Eq. (5) with $K = 1$, for which $E_T = (\frac{3}{4})^{1/2}$. The curve is a guide for the eye.

tions of the infinite-range model are relatively insensitive to N, and closely approximate the results presented here. To assess the effects of infinite-range coupling, we have numerically integrated Eq. (1) on a cubic lattice in three dimensions with nearest-neighbor coupling. The numerical solutions show hysteresis and switching,[21] though over a reduced range in E. For $N=216$ and $N=1000$ sites, the width of the hysteresis is approximately 20% and 15%, respectively, of the width predicted by the infinite-range model for the same values of N. Thus the qualitative behavior is similar to the mean-field theory, at least for finite sample sizes, but the thresholds are quantitatively different.

In summary, we have analyzed a dynamical system of many driven phases with random pinning and infinite-range coupling. The periodic coupling in the model gives rise to a first-order depinning transition, hysteresis, and switching between pinned and sliding states, and a time delay before the onset of sliding. These results demonstrate that some of the complex phenomena observed experimentally in strongly pinned charge-density-wave systems can be accounted for by a simple dynamical model.

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