Lower Critical Dimension for Populations of Oscillators with Randomly Distributed Frequencies: A Renormalization-Group Analysis

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(Received 13 July 1987; revised manuscript received 29 April 1988)

It is argued by way of a renormalization-group analysis that the lower critical dimension of macroscopic mutual entrainment in a class of populations of oscillators satisfies a certain inequality which is sensitive to the tail of the distribution of native frequencies. This result is supported in part by numerical simulations as well as a proof of the absence of long-range order in one dimension.

PACS numbers: 87.10.+e, 02.50.+s, 05.40.+j, 05.70.Fh

Recently, quite a few papers have been devoted to the dynamics of large populations of interacting self-oscillators with distributed native frequencies or winding numbers.[1-13] Such populations are known to model a variety of biological systems with rhythmic activity, e.g., diverse living organs such as the small intestine, swarms of fireflies, and so on.[1] The rhythmic activities exhibited by such systems may be identified with a macroscopic mutual entrainment (MME) of oscillatory elements where "macroscopic" means that the number of elements entrained to a common frequency is comparable to the population size N ($\gg 1$). Therefore, one of the important subjects in the study of large populations of oscillators is to search for conditions of the appearance of MME. Some new insights may be expected to come from such studies into the nature of temporal coherence that are very ubiquitous in the biological world. From a physicist's point of view, the onset of MME seems to resemble second-order phase transitions in equilibrium cooperative systems.[14] It is naturally important to examine to what extent such resemblance is true and to clarify what features of the onset of MME are new in comparison with equilibrium phase transitions. Seeking the conditions of MME may be interesting in this sense as well.

The oscillators treated here are dissipative dynamical systems exhibiting an attracting limit cycle whose period is expressed as $1/\Omega_j$ for the jth oscillator. The dynamics in large assemblies of such elements may be studied with a variety of models, of which the simplest is the following:

$$\dot{\theta}_j = \Omega_j + \sum_{i \in I_j} h_{ij}(\theta_i - \theta_j, \epsilon) \quad (1 \le j \le N), \tag{1}$$

where θ_j is the phase (divided by 2π) of the jth oscillator, and $\dot{\theta}_j \equiv d\theta_j/dt$. The second term in Eq. (1) comes from interactions of the jth oscillator with its "neighbors" whose set is denoted by I_j. The coupling function h_{ij} is periodic with period one in its first argument, and its second argument ϵ is a parameter which typically corresponds to the coupling strength. Though simple, this type of model is often used to analyze several concrete biological phenomena,[1,5,7] not only in studies with emphasis on the aspect of a phase transition.[1,2,8,9,13] [Recently, the discrete-time version of Eq. (1) has also been proposed and investigated to elucidate the dynamics in populations of doubly periodic oscillators.[10-12]] In this Letter, I am concerned with Eq. (1) subject to some restrictions on h_{ij} described below.

Given a large assembly of oscillators, it may depend on the type of interactions and the spatial dimensionality whether the assembly can exhibit MME or not. For a particular case of Eq. (1) such that each element is coupled to all others in an equal way, several authors demonstrated analytically and numerically the occurrence of MME beyond a certain threshold of the coupling strength.[1,2,6,9-12] Such models, however, may encounter an objection that they are not realistic enough. Interactions between elements in biological populations should typically be of finite range rather than of infinite range as is the case with the models. Recently, Sakaguchi, Shinomoto, and Kuramoto (SSK) went a step forward by considering a case of nearest-neighbor couplings with $h_{ij}(\theta,\epsilon) = \epsilon \sin 2\pi\theta$,[13] where the native frequencies Ω_j's were assumed to be independent random variables obeying a common normal law. On the basis of numerical simulations and some analytic arguments, they claimed that in such populations, MME can only appear when the spatial dimensionality d is larger than 2. Their arguments, however, do not appear fully convincing because of a serious approximation as well as an assumption of perfect entrainment invoked, as discussed in detail elsewhere.[15] The purpose of this Letter is to shed some light on this problem by way of a sort of real-space renormalization-group analysis.[16] For this, the couplings of the oscillators are assumed to be spatially finite ranged and bounded as long as ϵ is finite, satisfying

$$h_{ij}(-\psi,\epsilon) = -h_{ij}(\psi,\epsilon), \tag{2}$$

which is a generalization of the sinusoidal nearest-neighbor couplings as chosen by SSK[13] as well as some other authors.[5,7] Then, we show that the lower critical dimension of MME, denoted by d_c hereafter, obeys a certain inequality depending sensitively on the category of the distribution $f(\Omega)$ shared by natural frequencies that are independent random variables. This result is consistent not only with SSK's claim for the class of $f(\Omega)$ treated by them, but also with my own numerical simulations. At the end, I support the result in part by proving the absence of long-range order for a case of one-dimensional populations.

Now suppose that the whole d-dimensional lattice is divided into a set of hypercubes with an equal linear scale L (hence one cube contains $M \equiv L^d$ sites). For each of such hypercubes, let us imagine a single representative oscillator such that its phase ϕ is given by an average of the phases of all oscillators located in the hypercube or the block. I term such an imaginary oscillator a *block oscillator* which is analogous to the Kadanoff block spin.[16,17] The dynamics of the population of block oscillators may be determined by the following equations:

$$\dot{\phi}_k = \tilde{\Omega}_k + M^{-1} \sum_{(i,j)l} h_{ij}(\phi_l - \phi_k + \psi_{l,i} - \psi_{k,j}, \epsilon), \qquad (3)$$

where $\phi_k \equiv M^{-1} \sum_j \theta_j$ is the phase of the kth block oscillator and $\tilde{\Omega}_k \equiv M^{-1} \sum_j \Omega_j$, and the summation in Eq. (3) is taken for such pairs of (i,j) that the oscillators j and i belong to the kth block and one of its nearest-neighbor blocks, respectively (I assume $L \gg 1$). The residual phase ψ is defined through $\theta_j = \phi_m + \psi_{m,j}$ when the oscillator j is in the mth block. (Hereafter I put $N = \infty$ and omit the range of k, $1 \le k \le \infty$.)

In order to go a step forward, let us pay attention to the asymptotic behavior of the distribution function $f(\Omega)$ in the limit $|\Omega| \to \infty$ which will turn out to be crucial. Most generally, one may put

$$f(\Omega) \propto |\Omega|^{-\alpha-1} \quad (|\Omega| \gg 1), \qquad (4)$$

with $0 < \alpha \le 2$ unless $f(\Omega)$ has finite variance. The exponent α is important to classify distributions. For $f(\Omega)$ with a finite variance such as the normal law, I put $\alpha = 2$. I make use of the following fact: In the limit $n \to \infty$, the random variable $\hat{\Omega}_n \equiv (\sum_{i=1}^n \Omega_i - \gamma_n)/n^{1/\alpha}$ obeys one of the stable distributions whose characteristic function is essentially expressed as $\langle \exp(iz\hat{\Omega}\rangle = \exp(-|z|^\alpha)$, provided that the constant γ_n is appropriately chosen, where the angle brackets stand for an average.[18,19] [For example, if $\alpha = 2$, the stable distribution is the normal one. Actually, for $f(\Omega)$ with $\alpha = 2$ but without a variance, the denominator of $\hat{\Omega}_n$, $n^{1/2}$, has to be replaced by a constant $O((n \ln n)^{1/2})$. This fact, however, has no influence on the result below, Eq. (8).] Hence, I perform a set of transformations, $\tau = tM^{(1-\alpha)/\alpha}$ and $\hat{\phi}_k = \phi_k - (\gamma_M/M)t$, in Eq. (3) to obtain

$$d\hat{\phi}_k/d\tau = \hat{\Omega}_{M,k} + M^\beta \sum_{l \in J_k} \hat{h}_{lk}(\hat{\phi}_l - \hat{\phi}_k, \epsilon), \qquad (5)$$

where $\beta \equiv 1 - \alpha^{-1} - d^{-1}$ and J_k is the set of blocks nearest to the kth. The effective coupling function $\hat{h}_{lk}$ is defined by

$$\hat{h}_{lk}(\phi, \epsilon) = M^{(1-d)/d} \sum_{(i,j)} h_{ij}(\phi + \psi_{l,i} - \psi_{k,j}, \epsilon), \qquad (6)$$

which is at most $O(1)$ with respect to M.

The procedure of deriving Eq. (5) from Eq. (1) may be regarded as a sort of renormalization-group (RG) transformation. The renormalized evolution equations for the assembly of block oscillators reveal that in the limit $L \to \infty$, every system with the form of Eq. (1) is attracted to a *trivial fixed point* of the RG transformation:

$$d\hat{\phi}_k/d\tau = \hat{\Omega}_k, \qquad (7)$$

provided $\beta < 0$ where $\hat{\Omega}_k \equiv \lim_{M \to \infty} \hat{\Omega}_{M,k}$ obeys a stable distribution with the characteristic exponent α. Therefore, it follows that

$$d_c \ge \alpha/(\alpha - 1) \quad (1 < \alpha \le 2), \qquad (8)$$

and furthermore that for $0 < \alpha \le 1$, no MME (long-range order) arises in any finite number of dimensions. (For discrete-time systems, a similar analysis will be made elsewhere.[15])

Since SSK assumed the existence of a variance, their $f(\Omega)$ is in the class of $\alpha = 2$, for which our result (8) is consistent with their proposal. The above results reveal that large variations of the native frequency over the whole population, which are enhanced by decreasing α, have a drastic effect on the critical dimension d_c. Of particular interest may be the absence of MME in any finite number of dimensions for such a frequently used distribution as the Lorentzian[2,3,6,10–12] whose α is 1.

It is also suggested by the theory that in three dimensions, no MME occurs for $\alpha < \frac{3}{2}$. As a test of the theory, I have performed numerical simulations to check this prediction. For cubic lattices under periodic boundary conditions, I solved Eq. (1) by approximating θ_j with $[\theta_j((n+1)h) - \theta_j(nh)]/h$ $(h = 0.05)$ in the case of nearest-neighbor couplings of the form $h_{ij} = (\epsilon/2\pi) \times \sin 2\pi(\theta_i - \theta_j)$ and the distribution of Ω_j such that $f(\Omega) = a - \Omega^2$ $(|\Omega| \le 0.1)$ or $A|\Omega|^{-\alpha-1}$ $(|\Omega| \ge 0.1)$, where the constants a and A are fixed by the continuity of $f(\Omega)$ at $\Omega = 0.1$ as well as a normalization condition.[20] A sequence of uniformly random numbers was used as $\theta_j(0)$. Then, for $l = 80\,192$ steps after an initial 2000 ones discarded as a transient, I computed an average of θ_j, ω_j, for $1 \le j \le N$ to divide the whole population into clusters of mutual entrainment where $|\omega_i - \omega_j| < 1/2lh$ was used as a criterion of entrainment for oscillators i and j, following S^2K.[13] Figure 1 shows how the ratio of the largest cluster size to N, denoted by R, varies as ϵ is increased, where the data are averaged over 6, 3, and 3 samples of $\{\Omega_j\}$ for $N = 8^3$, 16^3, and 32^3, respectively. As is seen, for $\alpha = 1.7$ $(> \frac{3}{2})$, the data seem to converge for increasing N, suggesting a phase transition at a finite threshold in the infinite system, while for $\alpha = 1.3$ and 1.1 $(< \frac{3}{2})$, the curves tend to de-

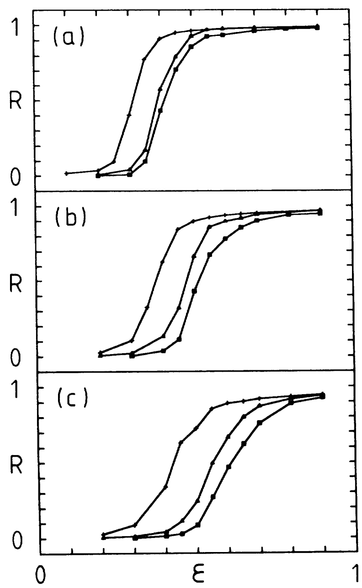

FIG. 1. R vs ϵ for (a) $\alpha=1.7$, (b) 1.3, (c) 1.1, where N is 8^3 (plusses: leftmost), 16^3 (triangles: middle), and 32^3 (squares: rightmost).

cay without convergence as N grows, in accordance with what is expected from the theory. Clearly, these results are still not sufficient: Further simulations for larger N as well as α closer to $\frac{3}{2}$ remain to be attempted.

Let us now focus on a particular one-dimensional (1D) version of Eq. (1) as follows:

$$\dot{\theta}_j=\Omega_j+h_j(\theta_{j+1}-\theta_j,\epsilon)+h_{j-1}(\theta_{j-1}-\theta_j,\epsilon) \quad (1\le j\le N), \tag{9}$$

with h_j satisfying the same prerequisites as before. According to the RG theory, long-range order cannot be expected in this type of system since $d_c\geq 2$, regardless of α. In what follows, I attempt to directly prove this for $f(\Omega)$ with a finite variance since otherwise $f(\Omega)$ possesses an infinite tail(s), rendering the proof easy (as is intuitively expected[13]). My basic assumption is the existence of the limit $\omega_j=\lim_{t\to\infty}[\theta_j(t)-\theta_j(0)]/t$ for all j. (However, uniqueness of the limit is not needed.) A sequence of neighboring oscillators is called a cluster when they share a value of ω. My goal is to show that the probability of finding a cluster with the size $O(N)$ vanishes for $N\to\infty$, as long as ϵ is finite.

Proof.—I begin with

$$\omega_j=\Omega_j+a_j-a_{j-1} \quad (1\le j\le N), \tag{10}$$

where a_j is a long-time average of $h_j(\theta_{j+1}-\theta_j,\epsilon)$. It is easy to derive from Eq. (10)

$$2a_{k+n/2}-a_{k+n}-a_k=\sum_{j=1}^{n}s_j\Omega_{k+j},$$

under the assumption that n oscillators, $k+1,\ldots,k+n$, form a cluster of mutual entrainment, where $s_j=-1$ $(1\le j\le n/2)$ or 1 $(n/2<j\le n)$. Therefore, a necessary condition for the appearance of such a cluster is $|\sum_{j=1}^{n}s_j\Omega_{k+j}|<4C$ (where C is a constant such that $|h_j|<C$) whose probability, P_n, may be given by

$$(2/\pi)^{1/2}(4C/\sigma)n^{-1/2} \tag{11}$$

when n is large, where σ^2 is the variance of $f(\Omega)$. Let us now consider a macroscopic cluster with the size βN $(0<\beta\le 1)$ which may be divided into a set of subclusters each of which has the length $\beta\sqrt{N}$. With this fact in mind, I arrive at $Q_N(\beta)<N(P_{\beta\sqrt{N}})^{\sqrt{N}}$, where $Q_N(\beta)$ is the probability for a cluster with the length βN to be found. By (11), $Q_N(\beta)$ vanishes as $N\to\infty$ QED.

In summary, my RG analysis based on the concept of "block oscillator" has led to the inequality (8) of the lower critical dimension for MME in the class of populations of oscillators as in Eq. (1) when $1<\alpha\le 2$, while it reveals the absence of MME in any finite number of dimensions for $0<\alpha\le 1$. My numerical simulations for 3D cubic lattices seem consistent with this result. A proof has also been given for the absence of MME in one dimension.

In this Letter, as a distribution of native frequencies, I have been concerned not only with distributions carrying a finite variance, but also with ones whose variance does not exist. At first sight, the latter appears to be much less relevant scientifically than the former. Recently, however, such "anomalous" distributions tend to become popular in connection with fractals.[19,21] They may be of practical importance in the context of population dynamics as well. If this is indeed the case, my result is very important because it reveals that the absence of long-range order in real systems (whose dimension is not larger than three) accompanied by a distribution with $\alpha<\frac{3}{2}$. This sensitivity to the tail of $f(\Omega)$ may be viewed as one of unique features of phase transitions in oscillator assemblies. My result is also expected to be useful in other areas beyond the biological context, e.g., in the study of coupled oscillating electric circuits. As stated earlier, however, further numerical simulations with greater accuracy are necessary to give more convincing support to the theory. Moreover, it should be asked what happens if the coupling functions do not meet the restrictions, e.g., (2). Needless to say, it is the value of d_c itself that has to be ultimately clarified. Although these remain to be done, I hope that this work will stimulate investigations in this field towards a complete understanding of the new type of critical phenomena.

Comments by H. Sakaguchi and S. Adachi are gratefully acknowledged.

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[1]A. T. Winfree, J. Theor. Biol. **16**, 15 (1967), and *The Geometry of Biological Time* (Springer-Verlag, New York,

1980).

[2]Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics,* edited by H. Araki, Lecture Notes in Physics Vol. 39 (Springer-Verlag, New York, 1975).

[3]Y. Aizawa, Prog. Theor. Phys. **56**, 703 (1976).

[4]Y. Yamaguchi, K. Kometani, and H. Shimizu, J. Stat. Phys. **26**, 719 (1981); Y. Yamaguchi and H. Shimizu, Physica (Amsterdam) **11D**, 212 (1984).

[5]A. H. Cohen, P. J. Holmes, and R. H. Rand, J. Math. Biol. **13**, 345 (1982).

[6]Y. Kuramoto, Suppl. Prog. Theor. Phys. **79**, 223 (1984).

[7]G. B. Ermentrout and N. Kopell, SIAM J. Math. Anal. **15**, 215 (1984); N. Kopell and G. B. Ermentrout, Commun. Pure Appl. Math. **39**, 623 (1986).

[8]G. B. Ermentrout, J. Math. Biol. **22**, 1 (1985).

[9]H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys. **76**, 576 (1986).

[10]H. Daido, Prog. Theor. Phys. **75**, 1460 (1986), and to be published.

[11]H. Daido, Prog. Theor. Phys. **77**, 622 (1987).

[12]H. Daido, J. Phys. A **20**, L629 (1987).

[13]H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987).

[14]H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971).

[15]H. Daido, to be published.

[16]S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).

[17]L. P. Kadanoff, Physics **2**, 263 (1966).

[18]B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Cambridge, 1954).

[19]H. Takayasu, *Hurakutaru* (Asakura, Tokyo, 1986), in Japanese.

[20]To realize such a distribution, I neglected the irrelevant part $|\Omega| > s$ with probability $1/2N$ to divide the remaining part into N intervals so that each of them has probability $1/N$. Then, the center values of these intervals were randomly assigned to the lattice sites, as native frequencies.

[21]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).