Quantum Localization for a Strongly Classically Chaotic System

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The classical stadium billiard, which is known to be ergodic and strongly mixing, is shown to have strongly localized quantum eigenstates which persist up to infinite energy, or alternately which survive the $\hbar \to 0$ limit. Consistent with the theorems of Schnirlman, Zelditch, and Colin de Verdiere, the states which we show are highly localized are collectively of zero measure (though there are infinitely many) as $E \to \infty$ or as $\hbar \to 0$. However, more relevant to the experimental world is the fact that such localized states make up a finite and calculable fraction of the quantum eigenstates at finite energy.

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In this paper we are concerned with the properties of individual eigenstates of classically chaotic systems,[1] from moderate energies up to the correspondence limit.[2-4] A strongly mixing (and thus ergodic) system, the Bunimovich stadium,[1] is shown to have infinitely many localized quantum eigenstates which persist up to an infinite energy, or alternately which survive the $\hbar \to 0$ limit. These states however are collectively of zero measure as $E \to \infty$ or as $\hbar \to 0$.

By demonstrating the existence of these exceptional quantum eigenstates, we disprove the hypothesis put forward by Berry[5]: "*Each* semiclassical eigenstate has a Wigner function concentrated on the region explored by a *typical* orbit over *infinite time*" (italics ours). It should be noted that the main spirit of Berry's hypothesis survives the findings reported here in the revised statement: "All (except a set of measure zero) semiclassical eigenstates have a Wigner function concentrated on the region explored by a typical orbit over infinite time." Colin de Verdiere[4] also conjectured that his proof of the ergodicity of all but an exceptional set of states could be tightened to include *every* state. This paper shows that this conjecture is incorrect.

What requirements does the correspondence principle place on individual eigenstates? The correspondence principle may be loosely stated as "quantum mechanics goes over into classical mechanics as the action becomes large compared to $\hbar$." More precisely, one needs to average, or *coarse grain*, the quantum data to remove the interference phenomena which remain in wave functions even in the high-action limit. Another problem, arising in time-dependent problems, is the order in which the limits $\hbar \to 0$ and $t \to \infty$ are taken ($\hbar \to 0$ first to recover classical mechanics). This implies that only properties of *groups* of eigenstates are needed to recover classical mechanics. Such averaging and grouping actually obliterates the details of individual eigenstates. The correspondence principle is thus unfortunately not a blueprint for the properties of individual eigenfunctions in the classical limit.

Semiclassical methods have proven very successful in revealing the properties of eigenstates (nodal structure, topology, amplitude structure, etc.) when the underlying classical mechanics is regular, or quasiperiodic.[5] Unfortunately, they are on much shakier grounds when the classical system possesses strong chaos. There is no semiclassical blueprint for the eigenstates of a strongly classically chaotic system. Progress has however been made on several fronts: (1) Statistical and averaged properties of eigenvalue spectra can be obtained from periodic orbits embedded in the chaos.[6-10] Some indication that individual eigenvalues may be found with the use of periodic orbits also exists.[6c] (2) Schnirlman,[2] Zelditch,[3] and Colin de Verdiere[4] have shown that eigenstates must conform to the domain of the *typical* classical trajectories of the same energy (i.e., the energy shell), except possibly for a set of measure zero (as $\hbar \to 0$) of such states. (3) A weak form of localization ("scars") was found by Heller[11] and McDonald and Kaufmann[12] and some scars were shown by Heller[11a] to be associated with certain unstable periodic orbits embedded in the chaotic dynamics. (4) Although less concerned with the properties of individual (quasi) energy eigenstates, work regarding driven systems[13-16] has led to significant new conclusions about quantum localization inside cantori, and Anderson localization[15] for driving terms leading to random impulses to the system. For excellent reviews, see Refs. 16 and 17. More detailed properties of individual eigenstates have remained largely in the domain of conjecture,[5] numerical experimentation,[11-14] and plausibility arguments.[18] Quantitative statements about asymptotic properties of specific eigenstates for classical chaotic systems have been lacking.

This paper is concerned with the properties of an exceptional set of states of the chaotic stadium billiard. Classically, the stadium billiard consists of free particle motion within a bounded region, subject to specular reflection off of the boundary. The boundary is stadium shaped, with two parallel straight edges of length $2\gamma r$, connected by semicircular endcaps of radius r. As long as the straight edges have finite length ($\gamma > 0$), the system is rigorously ergodic and mixing.

Quantum mechanically, to find the eigenstates one needs to solve the Helmholtz equation, $\nabla^2\Psi(\mathbf{x}) + k^2\Psi(\mathbf{x}) = 0$, subject to Dirichlet boundary conditions. In the earliest studies of the stadium eigenstates, McDonald and Kaufman[12] noted that even at fairly high energies, some of the eigenstates were very regular in appearance. These states were dubbed "bouncing-ball" states because they seemed to be associated with the classical short-time trapping of trajectories in bouncing vertically between the straight edges. Figure 1 displays the stadium boundary and one such bouncing-ball state. These states have their momentum confined to the vertical direction; they are not accessing the entire energy shell.

We provide a dynamical argument proving these nonstatistical states persist to arbitrarily large energies —they are *not* a low energy, large $\hbar$ phenomenon. We will also demonstrate that the fraction of the states that are localized in this way becomes vanishingly small as the energy rises to infinity.

If we create an arbitrary initial state with the requirement that it is quite localized in phase space, the mixing nature of the classical dynamics will require the state to breakup at an initial rate determined by the classical dynamics. We investigate nonstationary probe states that are localized in a region of phase space where such a breakup occurs very slowly. The initial disintegration rate is so slow that it severely constrains the spread in energy of the probe, and, likewise, strongly limits the number of eigenstates that can possibly have significant overlap with a probe. *If only a few eigenstates are needed to create a function which is highly localized in phase space, some or all of these eigenstates must themselves show a high degree of localization.* This will be the backbone of the argument for the localized eigenstates.

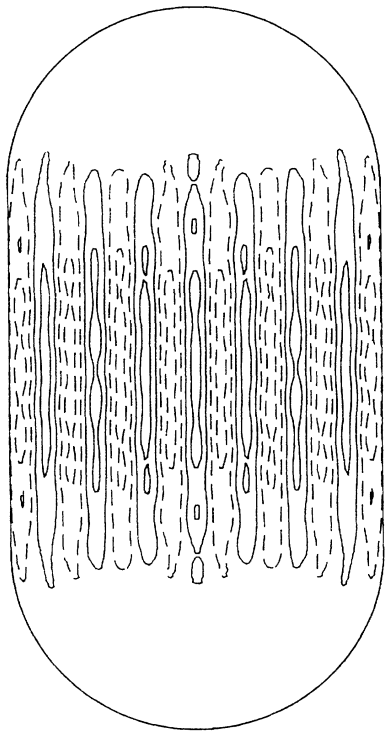

FIG. 1. Contour plot of a low-energy bouncing-ball state.

Specifically, we create a nonstationary probe state

$$\phi_n(\mathbf{x},0) = \exp(-x^2/2\sigma^2)\cos(p_y y)\,, \tag{1}$$

which initially corresponds to bouncing vertically between the parallel straight edges of the stadium. For convenience, we have set $\hbar = 1 = 2m$. This probe is spatially localized to the middle region of the stadium, but, more importantly, is strongly localized in momentum space to near the vertical direction.

The restriction that $\phi_n(\mathbf{x},0) = 0$ on the upper and lower straight edges quantizes the energy in the y direction, $E_{ny} = p_y^2$, to $E_{ny} = (n+\frac{1}{2})^2\pi^2/r^2$. We also want the Gaussian width, σ, to be as large as possible, yet small enough so that the amplitude of $\phi_n(\mathbf{x},0)$ is quite small in the endcaps. As σ is chosen by geometric considerations only (i.e., the length of the stadium), the momentum spread in the x direction, $\Delta p_x = 1/\sigma$, is constant, independent of p_y.

As long as there is essentially no amplitude for being in the endcaps, $\phi_n(\mathbf{x},t)$ will evolve according to

$$\phi_n(\mathbf{x},t) = g(x,t)e^{-iE_{ny}t}\cos(p_y y)\,, \tag{2}$$

where $g(x,t)$ is the free particle evolution of the Gaussian in Eq. (1),

$$g(x,t) = \left[\frac{1}{1+2it/\sigma^2}\right]^{1/2} \exp[-x^2/2(\sigma^2+2it)]\,.$$

Such an evolution of $\phi_n(\mathbf{x},t)$ assumes that any reflection of the wave off of the endcaps is negligible, and thus Eq. (2) is valid only up until a time t_E, before the spreading of $\phi_n(\mathbf{x},t)$ has caused it to reach the endcaps. More precisely, we define t_E by the time the probability that the particle is found between the straight edges has dropped to some fraction $\zeta < 1$. Thus, t_E is given by

$$\zeta = \frac{\int_{-\gamma r}^{\gamma r} dx \int_{-r}^{r} dy\, \phi_n^*(\mathbf{x},t_E)\phi_n(\mathbf{x},t_E)}{\int_{-\infty}^{\infty} dx \int_{-r}^{r} dy\, \phi^*(\mathbf{x},t_E)\phi_n(\mathbf{x},t_E)}\,.$$

These integrals yield the error function $\zeta = \mathrm{erf}(R)$, where $R = \gamma r/[\sigma(1+4t_E^2/\sigma^4)^{1/2}]$. Picking σ to maximize the time that we can use this approximate dynamics gives $\sigma = \gamma r/[\mathrm{erf}^{-1}(\zeta)\sqrt{2}]$, and $t_E = \sigma^2/2$. For the rest of this paper, any use of σ or t_E refers to these optimum values.

The most important property of t_E is its independence of the energy of the probe state. Energy is increased in $\phi_n(\mathbf{x},0)$ by increasing p_y, which does not play a part in the spreading into the endcaps. Until the endcaps are reached, the motion in the x direction is governed solely by spreading due to the uncertainty principle, $\Delta p_x = 1/\sigma$,

which is independent of E_{ny}.

By examining the dynamics of a nonstationary state, $|\phi_n(t)\rangle$, over a time T, we can deduce the range of energies involved in the evolution by studying the smoothed spectrum

$$S_T(\omega)=\frac{1}{2\pi}\int_{-T}^{T}dt\,\langle\phi_n(0)\,|\,\phi_n(t)\rangle e^{i\omega t}. \tag{3}$$

The $T\rightarrow\infty$ limit of $S_T(\omega)$ is simply the high-resolution spectrum of $|\phi_n(0)\rangle$,

$$S_\infty(\omega)=\sum_m |\langle\phi_n(0)\,|\,m\rangle|^2\delta(\omega-E_m),$$

where $|m\rangle$ denotes the mth eigenstate of the system, and E_m its corresponding energy. In keeping T finite, we merely look at the spectrum smoothed over an energy range of $2\pi/T$. Substituting the expression for $\phi_n(\mathbf{x},0)$ and $\phi_n(\mathbf{x},t)$ into Eq. (3), and performing the Fourier transform for as long as we are confident of the dynamics, $T=t_E$, we get

$$S_{t_E}(\omega)=\frac{1}{2\pi}\int_{-t_E}^{t_E}dt\left(\frac{1}{1+it/\sigma^2}\right)^{1/2}e^{i(\omega-E_{ny})t}. \tag{4}$$

The sole effect of E_{ny} in this expression is to shift $S_{t_E}(\omega)$ in energy, leaving its shape and breadth alone. A plot of $S_{t_E}(\omega)$ is displayed in Fig. 2. Since $t_E/\sigma^2=\frac{1}{2}$, the function $1/(1+it/\sigma^2)^{1/2}$ is roughly constant over the range of integration, indicating that the only frequencies that yield an appreciable amplitude for $S_{t_E}(\omega)$ are those that are within $\Delta\omega=\pi/t_E$ on either side of E_{ny}. This estimate corresponds to the central hump seen in Fig. 2.

By estimating the energy-level spacing, we can count how many eigenstates the central hump of Fig. 2 spans. For two-dimensional billiards problems, the average energy-level spacing, ΔE, is *independent* of energy. The appropriate ΔE for our problem [considering that $\phi_n(\mathbf{x},0)$ accesses eigenstates of only one symmetry class] is given by[19] $\Delta E=4\pi/r^2(\gamma+\pi/4)$. The approximate number of eigenstates under this band, $\mathcal{N}$, is then

$$\mathcal{N}=\frac{2\Delta\omega}{\Delta E}=\frac{2(\gamma+\pi/4)}{\gamma^2}[\mathrm{erf}^{-1}(\zeta)]^2. \tag{5}$$

To get an idea of the number of states this involves, we will use the most frequently studied stadium, $\gamma=1$, and set our cutoff parameter to $\zeta=0.98$. These values give $\mathcal{N}\approx 10$. Ten states must "share" the probe state. Under the assumption that they do so democratically, each of these ten contributing eigenstates have one tenth of their probability concentrated in the region of phase space covered by $\phi_n(\mathbf{x},0)$. The fraction of the energy shell that this encompasses is roughly $\Delta p/p=\lambda/\sigma$, where λ is the de Broglie wavelength. In the semiclassical limit, $\lambda\rightarrow 0$, this fraction vanishes! These states are thus not *uniformly* concentrated over the entire energy shell, but instead show a high degree of localization. If these ten states do not share the vertical bounce state equally, then some one or few among them must have even more localization to the vertical bounce region of phase space.

The estimate of the dilution of the nonstationary state into a maximum of ten eigenstates is independent of the energy. We can create the probe states with narrow energy bands around $E_{ny}=(n+\frac{1}{2})^2\pi^2/r^2,(n+\frac{3}{2})^2\pi^2/r^2,\ldots$. These energies grow farther apart with n, yet the width of the bands and the density of states do not change. Thus the clumps of ten (or less) states that are affected by the bouncing-ball motion are spaced farther and farther apart as energy increases, becoming a vanishing fraction of the total in the high-energy limit. Thus, they are of measure zero. However, they are also more severely localized as $E\rightarrow\infty$, since the fraction of the energy shell which the initial state represents vanishes as $E\rightarrow\infty$.

The above localization arguments depend *only* on $\mathcal{N}$ being independent of the energy of the probe state, and *not* on a specific choice of ζ. ζ affects only the degree of localization that one would expect to find. However, as we have no unambiguous method for choosing a particular value of ζ, these arguments do not tell us whether $\mathcal{N}$ is ten states or one. Numerically, up to about the 2500th state, it appears that a single eigenstate is heavily responsible for the vertical bounce character of the nonstationary state, as seen in Fig. 3. At least at these energies, the estimate of ten states is quite conservative.

Finally, we note that the deviation between the true evolution of $\phi_n(\mathbf{x},0)$ and the approximation of Eq. (2) is a measure of the probability that has actually reached the endcaps. As we do not know now to accurately handle the reflection from the endcaps, we halt the dynamics at a time t_E that "too much" probability had been reflected, and thus also limited the resolution with which we could study the spectrum. We can, however, study the spectrum at higher resolution and rid ourselves of the ζ ambiguity with a simple, physical argument. In the classical stadium, the endcaps serve to randomize the

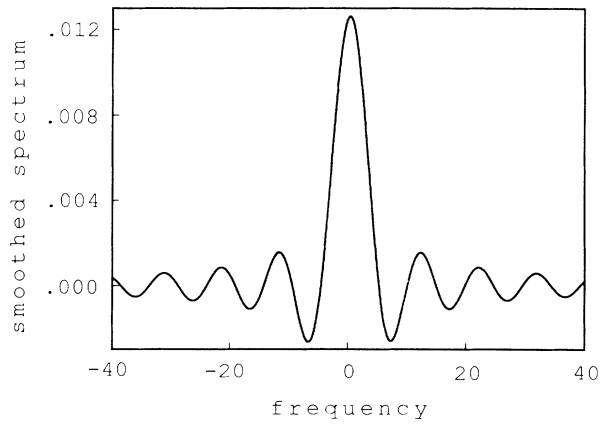

FIG. 2. Plot of the smoothed spectrum, $S_{t_E}(\omega)$, for the probe state in Eq. (1) vs $(\omega-E_{ny})/\Delta E$.

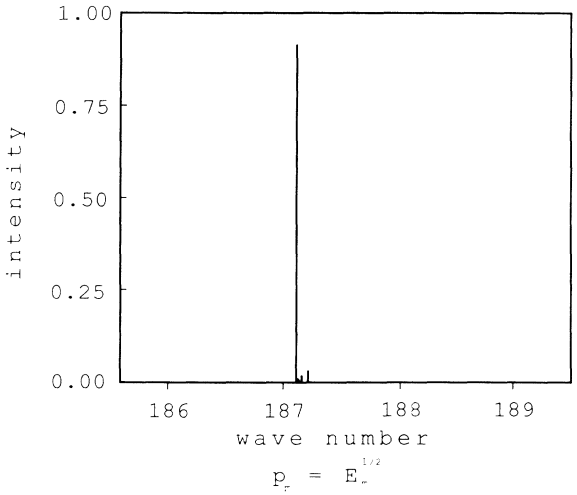

FIG. 3. The high-resolution spectrum, $|\langle\phi_n|E_m\rangle|^2$, for a probe state chosen to have 58 horizontal nodes ($n=29$). There are 127 eigenstates with energies which lie in the range shown; however, only 1 had large overlap with the probe.

horizontal momentum of nearly bouncing-ball orbits.[20] Following suit, we break up $\phi_n(\mathbf{x},t)$ into two parts—a "coherent" part of which has not yet reached the end-caps, and an "incoherent" part which has. If we eliminate the incoherent part from the autocorrelation function, we are again being conservative, since any recurrence which we ignore would only sharpen the spectrum. A recalculation of $\mathcal{N}$ under these assumptions[21] gives $\mathcal{N}\approx 1$, in agreement with Fig. 3.

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[1a]L. A. Bunimovich, Funct. Anal. Appl. **8**, 254 (1974).
[1b]L. A. Bunimovich, Commun. Math. Phys. **65**, 295 (1979).
[2]A. I. Schnirlman, Usp. Mat. Nauk. **29**, 181 (1974).
[3]S. Zelditch, Duke Math. J. **55**, 919 (1987).
[4]Y. Colin de Verdiere, Commun. Math. Phys. **102**, 497 (1985).
[5]M. V. Berry, in *Chaotic Behavior of Deterministic Systems,* edited by G. Iooss, R. H. G. Helleman, and R. Stora (North-Holland, Amsterdam, 1983), p. 172.
[6a]M. C. Gutzwiller, J. Math. Phys. **11**, 1791 (1970).
[6b]M. C. Gutzwiller, J. Math. Phys. **12**, 343 (1971).
[6c]M. C. Gutzwiller, Physica (Amsterdam) D **5**, 183 (1982).
[7a]M. V. Berry and K. E. Mount, Rep. Prog. Phys. **35**, 315 (1972).
[7b]M. V. Berry and M. Tabor, Proc. Roy. Soc. London A **349**, 101 (1977).
[7c]M. V. Berry and M. Tabor, Proc. Roy. Soc. London A **356**, 375 (1977).
[7d]M. V. Berry, Proc. Roy. Soc. London A **400**, 2 9 (1985).
[8]J. H. Hannay and A. M. Orzorio de Almedia J. Phys. A **17**, 3429 (1984).
[9]N. L. Balazs and A. Voros, Phys. Rep. **143**, 109 (1986).
[10a]D. Wintgen, Phys. Rev. Lett. **58**, 1589 (1987).
[10b]D. Wintgen and H. Friedrich, Phys. Rev. A **36**, 131 (1987).
[11a]E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).
[11b]P. W. O'Connor, J. N. Gehlen, and E. J. Heller, Phys. Rev. Lett. **58**, 1296 (1987).
[12]S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. **42**, 1189 (1979).
[13]T. Geisel, G. Radons, and J. Rubner, Phys. Rev. Lett. **57**, 2883 (1986).
[14]R. C. Brown and R. E. Wyatt, Phys. Rev. Lett. **57**, 1 (1986).
[15]S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).
[16]G. Radons, T. Geisel, and J. Rubner, to be published.
[17]B. Eckhardt, Phys. Rep. **163**, 205 (1988).
[18]Y. Y. Bai, G. Hose, K. Stefanski, and H. S. Taylor, Phys. Rev. A **31**, 2821 (1985). These authors also consider bouncing-ball states, and describe a plausible mechanism for their existence, based on adiabatic arguments.
[19]The energy-level spacing was found from the crudest form of the Weyl rules, by which one counts the number of states supported by a phase-space volume by assigning one state per volume h^D, where D is the number of degrees of freedom. See also R. Balian and C. Bloch, Ann. Phys. (N.Y.) **69**, 76 (1972).
[20]K. C. Lee, Phys. Rev. Lett. **60**, 1991 (1988).
[21]P. W. O'Connor and E. J. Heller, to be published.