Approximation Scheme for Constructing a Clumpy Universe in General Relativity

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We shall develop an approximation scheme to construct the metric representing a realistic clumpy universe in general relativity. Spatial averaging is used to express the fact that the expansion of the universe is generated collectively by the clumps of matter. The dynamics of the clumps is treated by the post-Newtonian–type approximation. The scheme allows one to calculate the back reaction due to the growth of clumps on the expansion and vice versa. An expression for the deviation from homogeneous and isotropic expansion is derived in terms of inhomogeneities of the gravitational field generated by clumps. It should also be used as the correct interpretation of the observation of gravitational lenses.

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The standard Friedman-Robertson-Walker (FRW) universe models assuming exact homogeneity, isotropy, and a hot "big bang" have had tremendous succes in describing the early universe.[1] FRW models are firmly supported by the observation of the thermal spectrum and isotropies in the cosmic microwave background.[2]

On the other hand, the applicability of FRW models at the present epoch has sometimes been questioned.[3] On scales below ~100 Mpc, the observed universe is neither homogeneous nor isotropic, but rather clumpy. Recent progress in observational techniques has revealed amazing large-scale structures, such as the presence of large voids and superclusters.[4]

In spite of extensive theoretical efforts to explain such large-scale as well as small-scale structures in the universe, the formation of these structures still remains a mystery.[5] It is hoped that further information, such as more complete observation of the three-dimensional distribution of galaxies at high red shifts, will improve the situation.

These observations would give important information on large-scale spacetime inhomogeneities because the inhomogeneities influence the propagation of light. Thus the information provided by observations would be useful only if we had a reliable description of inhomogeneities in the expanding universe and of the propagation of light in such an inhomogeneous universe. As far as the latter is concerned, the geometric-optics approximation based on general relativity[6] offers a sufficiently accurate description in the cosmological context. However, it seems that no satisfactory approximation for describing an inhomogeneous universe is available except in the linearized case in which the density contrast is supposed to be small.[7] An approximation in which the density contrast is much larger than unity is particularly needed for the correct interpretation of the observation of gravitational lenses.[8] The linearized theory is clearly not sufficient for such a purpose.

The aim of the present work is to develop an approximation scheme for constructing the metric of fully inhomogeneous (clumpy) universe models within the framework of general relativity. The point here is to realize that the metric perturbation can remain small even when the density contrast is much larger than unity. The solar system is a typical example of this. We have a very powerful method of calculating relativistic effects for such systems, namely, the post-Newtonian approximation.[9] The size of the metric perturbation and that of the density contrast are *independent* of each other in the exact theory (models of an isolated star surrounded by a vacuum field) as well as in post-Newtonian approximations. The metric may often be computed as a function of the source from a linear approximation to the solution of the field equation, whereas motions have to be computed from the nonlinear equation; otherwise the first-order metric does not approximate a solution of Einstein's equation over dynamical time scales (say, periods). Moreover, in linearized approximation the mutual gravitational interaction between clumps is not taken into account. For this reason it is also necessary to have the post-Newtonian–type approximation in cosmology.

There have been some studies on self-gravitating fluids[10] and post-Newtonian approximation[11] in an expanding universe. In these treatments the expansion law is prescribed and fixed once and for all in spite of the fact that the expansion is generated collectively by the clumps of matter. We will make use of a simple spatial averaging by assuming spatial periodicity in the initial data to express this fact. Then the global equations for the universe expansion and the equations for local inhomogeneities (clumps) become coupled equations and they have to be solved simultaneously. Thus the back reaction of the development and motion of the clumps on the expansion of the universe, and vice versa, may be calculated. Since Einstein's equation is nonlinear, any averaging process is far from trivial in general. However, we are not treating here the most general situation in the exact theory. Rather we are developing an approximation scheme in a cosmological situation. In this case I give an argument that all complexities which might appear in the full theory are, in fact, of higher order and may be neglected.

We first make the following *Ansatz* for the metric:

$$g_{\mu\nu}=a^2(\tilde{g}^{(b)}_{\mu\nu}+h_{\mu\nu})=a^2\tilde{g}_{\mu\nu}, \tag{1}$$

where a is supposed to describe the global expansion and is assumed to be a function of time only. $\tilde{g}^{(b)}$ is given by

$$\tilde{g}^{(b)}_{\mu\nu}=-d\eta^2+d\Omega_3^2(k), \tag{2}$$

where $d\Omega_3^2(k)$ is the standard metric on S^3 if $k=1$ and on R^3 if $k=0$ or -1. The $h_{\mu\nu}$'s are supposed to describe local inhomogeneities and are assumed to be small. As explained above, this does not imply that the density contrast is small. The above metric is the standard FRW one expressed by the conformally rescaled time η when $h_{\mu\nu}$ vanishes. This does not mean that the zeroth-order spacetime is necessarily one of the FRW spacetimes, for that depends upon the approximation that one chooses. In the linearized approximation, the zeroth-order spacetime is taken to be an FRW spacetime. On the other hand, if one employs the post-Newtonian approximation, the zeroth-order spacetime is the conformally transformed Newtonian spacetime which is a four-dimensional manifold with degenerate metric.[12] For the moment we shall retain a more general level and derive the formal equations under the assumption that h is small and that the scale on which h varies is small compared to that of a and $\tilde{g}^{(b)}$. These assumptions shall be made more precise in a moment.

In the following calculation it is convenient to use the trace-reversed perturbation defined by

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\tilde{g}^{(b)}_{\mu\nu}h, \tag{3}$$

and to work in the "harmonic gauge"

$$\bar{h}^{\mu\nu}{}_{|\nu}=0, \tag{4}$$

where $h=\tilde{g}^{(b)\mu\nu}h_{\mu\nu}$. The indices on h are shifted by $\tilde{g}^{(b)}$ and the bar $(|)$ indicates the covariant derivative with respect to $\tilde{g}^{(b)}$.

The above *Ansatz* for the metric is used to calculate the Einstein equation as follows:

$$(a'/a)^2(4\tilde{g}^{(b)\mu\eta}\tilde{g}^{(b)\nu\eta}-\tilde{g}^{(b)\eta\eta}\tilde{g}^{(b)\mu\nu})-2(a''/a)(\tilde{g}^{(b)\mu\eta}\tilde{g}^{(b)\nu\eta}-\tilde{g}^{(b)\eta\eta}\tilde{g}^{(b)\mu\nu})+A^{\mu\nu}$$
$$+(a'/a)(2\bar{h}^{\eta(\mu|\nu)}-\bar{h}^{\mu\nu|\eta}-\tilde{g}^{(b)\eta(\mu}\bar{h}^{|\nu)}+\tfrac{1}{2}\tilde{g}^{(b)\mu\nu}\bar{h}^{|\eta})-\tfrac{1}{2}\bar{h}^{\mu\nu|\rho}{}_{|\rho}=8\pi G\tau^{\mu\nu}, \tag{5}$$

where $A^{\mu\nu}$ is the background spatial curvature term given by $A^{\eta\eta}=-3k\tilde{g}^{(b)\eta\eta}=3k$, $A^{ij}=-k\tilde{g}^{(b)ij}$, and $A^{\mu k}=0$. $\tau^{\mu\nu}=a^4T^{\mu\nu}+t^{\mu\nu}$ and $t^{\mu\nu}$ consists of terms quadratic in $\bar{h}$ and may be regarded as a gravitational stress-energy pseudotensor.[13] In the above calculation terms like $\bar{h}^3$, $(a'/a)\bar{h}^2$, and $(a''/a)\bar{h}$ have been neglected. Since we have retained terms like $\bar{h}_{\rho\sigma}{}^{|\mu}\bar{h}^{\rho\sigma|\nu}$ in $t^{\mu\nu}$, this means that we are treating here the metric perturbations whose amplitudes are larger than a''/a or $(a'/a)^2$. This is the condition that the self-gravity of clumps has a more important effect on their dynamics than the expansion of the universe. In order to make this point more clear, we shall introduce two independent small parameters ϵ and κ. The ϵ is associated with the size (amplitude) of h, $h=O(\epsilon^2)$; and the κ is associated with the ratio between the scale of the variation l of h and that L of a and $\tilde{g}^{(b)}$; $\kappa=l/L$. Then the above condition implies $\epsilon>\kappa$. For example, if we take a cluster of galaxies whose size is about 1 Mpc, then κ will be $\sim$(1 Mpc)/(10^4 Mpc)$\sim 10^{-4}$, where $L\sim 10^4$ Mpc is the present horizon size. Thus we are interested in the metric perturbations whose amplitudes [$\sim O(\epsilon^2)$] are larger than 10^{-8}. In the case of superclusters whose size is of the order of 50 Mpc, the amplitude of the metric perturbation might not satisfy the above condition and one has to take into account terms like $(a''/a)\bar{h}$. The inclusion of such terms allows us also to treat the transition from the linear to nonlinear regime. It will be shown in a future publication that the inclusion of such terms will not cause any difficulty in the present scheme.[14] As long as we consider scales less than superclusters, we can safely neglect these terms.

Equations of motion are derived from the conservation of the stress-energy tensor as usual,

$$T^{\mu\nu}{}_{|\nu}+(a'/a)(6T^{\eta\mu}-\tilde{g}^{(b)\eta\mu}T+\bar{h}^{\eta\mu}T-\tilde{g}^{(b)\eta\mu}\bar{h}_{\rho\sigma}T^{\rho\sigma})+(\bar{h}^{\mu}{}_{\rho|\sigma}-\tfrac{1}{2}\bar{h}_{\rho\sigma}{}^{|\mu})T^{\rho\sigma}-(\bar{h}_{|\rho}T^{\rho\mu}-\tfrac{1}{4}\bar{h}^{|\mu}T)=0, \tag{6}$$

where terms of order $\bar{h}^2T$ have been neglected.

Now we take the spatial average of Eq. (5) assuming spatial periodicity of the material initial data as well as of the free data for the gravitational field. The spatial average over a volume V is defined as usual,

$$\langle Q\rangle=V^{-1}\int_V Q\,dV, \tag{7}$$

where dV is the invariant volume element in the background space. It should be noted that the only property that we shall use in the calculation below is $\langle Q_{|i}\rangle=0$. Spatial periodicity does imply this. However, if the averaging volume is large enough and the perturbations are randomly distributed, then this quantity $\langle Q_{|i}\rangle$ is almost always negligibly small.[15] Thus the equations derived in this paper may hold for more general aperiodic perturbations. The result is given by

$$\left(\frac{a'}{a}\right)^2=\frac{8\pi G}{3}\langle\tau^{\eta\eta}\rangle-k\,, \tag{8}$$

$$\frac{a''}{a}=\frac{4\pi G}{3}\langle\tau^{\eta\eta}-\tau^k_k\rangle-k\,, \tag{9}$$

$$a^{-2}\{a^2\langle\bar{h}^{ij}\rangle|_\eta\}|_\eta=16\pi G\langle\hat{\tau}^{ij}\rangle\,, \tag{10}$$

where I have defined the spatial trace-free part of τ^{ij}: $\hat{\tau}^{ij}=\tau^{ij}-\frac{1}{3}\tilde{g}^{(b)ij}\tau^k_k$. Note that the average of the time space components of Eq. (5) gives $\langle\tau^{\eta i}\rangle=0$. In deriving the above expression use has been made of the facts that $\langle\bar{h}_{\eta\eta}\rangle=\langle\bar{h}_{\eta i}\rangle=0$ and $\langle\bar{h}^k_k\rangle=0$. The spatial average of the gauge condition implies that $\langle\bar{h}_{\eta\eta}\rangle$ and $\langle\bar{h}_{\eta i}\rangle$ are constants and thus we may eliminate them by a suitable coordinate transformation. The third equality comes from the fact that $\langle\bar{h}^k_k\rangle$ expresses an additional isotropic expansion and the effect is absorbed into the scale factor by an appropriate redefinition of the time variable and scale factor. Since the constant part in $\langle\bar{h}^{ij}\rangle$ has no physical meaning, we may put that part equal to zero and then immediately integrate Eq. (10) to get

$$\langle\bar{h}^{ij}\rangle(\eta)=\int_{\eta_0}^{\eta}\frac{d\eta'}{a^2(\eta')}\langle\bar{h}^{ij}\rangle|_\eta(\eta_0)+16\pi G\int_{\eta_0}^{\eta}\frac{d\eta'}{a^2(\eta')}\int_{\eta_0}^{\eta'}d\eta''\,a^2(\eta'')\langle\hat{\tau}^{ij}\rangle\,. \tag{11}$$

Since the spatial average of the line element takes the following form,

$$\langle ds^2\rangle=a^2\{-d\eta^2+(\delta_{ij}+\langle\bar{h}_{ij}\rangle)dx^i dx^j\}\,, \tag{12}$$

Eq. (11) is the expression for the deviation from isotropic expansion in terms of the inhomogeneities. Thus global spacetime expands anisotropically except if $\langle\bar{h}^{ij}\rangle$ vanishes identically. A sufficient condition for global isotropic expansion (statistically homogeneous and isotropic model) is given by

$$\langle\bar{h}^{ij}\rangle_{,\eta}(\eta_0)\equiv0 \quad\text{and}\quad \langle\hat{\tau}^{ij}\rangle\equiv0\,. \tag{13}$$

The equations which determine the evolution of the local inhomogeneities are derived by substituting the above equations back into the original Eq. (5). These are

$$\Box\bar{h}^{\eta\eta}=-16\pi G(\tau^{\eta\eta}-\langle\tau^{\eta\eta}\rangle)+(a'/a)(\bar{h}^{\eta i}|_i-\bar{h}^k{}_k|_\eta)\,, \tag{14}$$

$$\Box\bar{h}^{\eta i}=-16\pi G\tau^{\eta i}+(a'/a)(\bar{h}^{\eta\eta|i}+\bar{h}_k{}^{k|i})\,, \tag{15}$$

$$\Box\hat{\bar{h}}^{ij}=-16\pi G\hat{\tau}^{ij}+(a'/a)[2\hat{\bar{h}}^{ij}|_\eta+4(\bar{h}^{\eta(i|j)}-\tfrac{1}{3}\tilde{g}^{(b)ij}\bar{h}^{\eta k}|_k)]\,, \tag{16}$$

$$\Box\bar{h}^k{}_k=-16\pi G(\tau^k{}_k-\langle\tau^k{}_k\rangle)+(a'/a)(\bar{h}^k{}_k|_\eta-\bar{h}^{\eta k}|_k)\,, \tag{17}$$

where $\hat{\bar{h}}^{ij}=\bar{h}^{ij}-\frac{1}{3}\tilde{g}^{(b)ij}\bar{h}^k{}_k$ and $\Box\bar{h}^{\mu\nu}=\bar{h}^{\mu\nu|\rho}|_\rho$. These equations are supplemented by the equations of motion given by Eq. (6).

Up to this point we have not employed any particular approximation scheme except that we have been neglecting terms of order $\bar{h}^3$, $(a'/a)\bar{h}^2$, and $(a''/a)^2\bar{h}$. What kind of approximation one chooses for solving the equations depends upon the kind of physical situation that one has in mind. The situation I wish to describe here is the universe with material clumps of various scales interacting gravitationally with each other. The density contrast between these clumps and the mean density is much larger than unity. The typical peculiar velocities of the clumps are much less than the speed of light. If we ignore the effect of the cosmic expansion, these clumps are well described by a Newtonian theory. Thus it is natural to solve the local equations (14)-(17) by means of a post-Newtonian approximation. We shall therefore employ a post-Newtonian approximation and demonstrate how one can solve the above equations perturbatively up to the first nontrivial order, i.e., up to the order at which the first nonvanishing effect of inhomogeneities upon the expansion appears. For simplicity we shall take the $k=0$ case and a perfect fluid as an example of the material source:

$$T^{\mu\nu}=[\rho+p(\rho)]u^\mu u^\nu+p(\rho)g^{\mu\nu}\,, \tag{18}$$

where ρ is the density, p the pressure, and u^μ is the four-velocity.

The standard method of calculating the post-Newtonian approximation shows that the $T^{\mu\nu}$ in the lowest order are as follows:

$$T^{\eta\eta}=a^{-2}\rho,\quad T^{\eta i}=a^{-2}\rho\tilde{v}^i\,,\qquad T^{ij}=a^{-2}(\rho\tilde{v}^i\tilde{v}^j+p\delta^{ij})\,, \tag{19}$$

where $\tilde{v}^i=dx^i/d\eta$. Using these expressions, one obtains the lowest-order equations for the metric perturbation:

$$\Delta\bar{h}^{\eta\eta}=-16\pi Ga^2(\rho-\rho_b)\,. \tag{20}$$

This is the lowest-order approximation to Eq. (14). Other equations [(15)-(17)] are not needed because they contribute to the second post-Newtonian order.[9] The ρ_b is the mean density defined by $\rho_b=V^{-1}\int_V\rho\,d^3x$. We use these expressions to calculate the gravitational stress-energy pseudotensor in the lowest order:

$$t^{\eta\eta}=-(8\pi G)^{-1}a^4[4\phi\Delta\phi+3(\nabla\phi)^2]\,,\qquad t^{ij}=(8\pi G)^{-1}a^4\{-2\phi^{,i}\phi^{,j}+4\phi\phi^{,ij}+\delta^{ij}[4\phi\Delta\phi+3(\nabla\phi)^2]\}\,, \tag{21}$$

where ϕ is defined by $\bar{h}^{\eta\eta}=4a^2\phi$. One may see now that the effect of inhomogeneities expressed by the "Newtonian" potential ϕ appears at the first post-Newtonian order. Thus we have to calculate $T^{\eta\eta}$ up to the first post-Newtonian order for consistency. At the end we have the following expressions for the total effective stress-energy pseudotensor:

$$\tau^{\eta\eta}=a^2\rho+[a^2\rho(\tilde{v}^2+2a^2\phi)-(8\pi G)^{-1}a^4(4\phi\Delta\phi+3(\nabla\phi)^2)]\,, \tag{22}$$

$$\tau^{ij}=a^2(\rho\tilde{v}^i\tilde{v}^j+p\delta^{ij})+(8\pi G)^{-1}a^4[-2\phi^{,i}\phi^{,j}-4\phi\phi^{,ij}+\delta^{ij}(4\phi\Delta\phi+3(\nabla\phi)^2)]\,. \tag{23}$$

The averaging gives

$$\langle\tau^{\eta\eta}\rangle=a^2\rho_b+a^2\langle\rho\tilde{v}^2\rangle+a^4(5/8\pi G)\langle(\nabla\phi)^2\rangle\,, \tag{24}$$

$$\langle\tau^{ij}\rangle=a^2\langle\rho\tilde{v}^i\tilde{v}^j+p\delta^{ij}\rangle+a^4(1/8\pi G)\langle 2\phi^{,i}\phi^{,j}-\delta^{ij}(\nabla\phi)^2\rangle\,, \tag{25}$$

where use has been made of the equation for ϕ, namely, $\Delta\phi=-4\pi G(\rho-\rho_b)$.

These averaged sources are used to calculate the global expansion once we know the time evolution of the material quantities ρ and $\tilde{v}^i$ as functionals of the scale factor a. These are calculated from the equations of motion (6). For the calculation up to the first nontrivial order that we are interested in here, the knowledge of these quantities at the Newtonian order is sufficient. The equations (6) become up to that order

$$\rho_{,\eta}+3(a'/a)\rho+(\rho\tilde{v}^i)_{,i}=0\,, \tag{26}$$

$$\tilde{v}^i{}_{,\eta}+(a'/a)\tilde{v}^i+\tilde{v}^j\tilde{v}^i{}_{,j}+\rho^{-1}p^{,i}=a^2\phi^{,i}\,. \tag{27}$$

Equations (8)-(10), (20), and (24)-(27) are the coupled equations to be solved simultaneously.

We have developed an approximation scheme to construct a model of a fully inhomogeneous universe based on the post-Newtonian approximation within the framework of general relativity. By using a simple spatial averaging procedure, this scheme allows one to calculate the back reaction due to the growth of local inhomogeneities on the cosmic expansion and vice versa.

This scheme may also be used for the correct interpretation of observations of gravitational lenses. So far rather crude descriptions of inhomogeneities have been used in the study of light propagation.[16] The absence of a realistic metric of an inhomogeneous (clumpy) universe is the major ambiguity in the theory of gravitational lenses. Once such a metric is known, the problem becomes a simple integration of the focusing equation of the light bundle. The present scheme provides an approximation to the metric of such a clumpy universe and thus it is expected to play a fundamental role in the correct interpretation of observations to reveal the nature of the image source as well as the lensing object.

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