Dynamical Symmetry Breaking and Quantum Nonintegrability

Wei-Min Zhang,[(1)] Craig C. Martens,[(2)] Da Hsuan Feng,[(1)] and Jian-Min Yuan[(1,2)]

[(1)]*Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104*

[(2)]*Department of Chemistry, University of Pennsylvania, Philadelphia, Pennsylvania 19104*

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In this Letter, we describe the connection between the classical concepts of nonintegrability and chaos and the quantum concept of dynamical symmetry breaking. The existence of unbroken dynamical symmetry implies integrability of the mean-field motion in a quantum phase space, defined as a symplectic coherent-state parametrization of the coset space of the overall dynamical group. Broken dynamical symmetry leads to nonintegrability, and thus chaotic solutions to Hamilton's equations in the quantum phase space. We illustrate the general ideas with results obtained for a model of two coupled spins.

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The study of the quantum manifestations of classical chaos has attracted considerable interest in recent years.[1] However, unlike the well-defined classical concept,[2,3] there exists no generally accepted definition for quantum chaos. In this Letter, we address this problem by establishing a relation between the concepts of dynamical symmetry and quantum integrability. This is accomplished by introducing an unambiguous and general definition of the phase space of quantum systems in terms of the mean-field motion in the coherent-state representation.

A classical mechanical system with N degrees of freedom is integrable if a set of N constants of the motion with mutually vanishing Poisson brackets (i.e., in involution) can be found.[2,3] An analogous situation occurs in quantum mechanics if a complete set of N commuting operators can be found which also commute with the Hamiltonian of the system. This can be explicitly realized for a quantum system having a dynamical group, and a precise relation between integrability and dynamical symmetry can be made.

A quantum system has a dynamical group G if the Hamiltonian of the system is a function in the generators of G. In this case, the corresponding Hilbert space can be decomposed into a direct sum of the irreducible carrier spaces of G. Therefore, for such a system the discussion of the physical properties can be restricted to one of the irreducible carrier spaces (i.e., to a single irreducible representation). Furthermore, the system has a *dynamical symmetry* if the Hamiltonian is solely a function of the *Casimir operators* of a particular group chain of G.[4,5] Explicitly, we have

$$H=f(C_\alpha^i),\quad i=1,2,\ldots,s\,, \tag{1}$$

for a given α, where C_α^i is the Casimir operator belonging to the ith subgroup of the αth group chain of G:

$$G\supset\begin{cases} G_1^s\supset G_1^{s-1}\supset\cdots\supset G_1^1\,,\\ \cdots\\ G_\alpha^s\supset G_\alpha^{s-1}\supset\cdots\supset G_\alpha^1\,,\\ \cdots\\ G_\gamma^s\supset G_\gamma^{s-1}\supset\cdots\supset G_\gamma^1\,,\end{cases} \tag{2}$$

for G with γ subgroup chains. This implies that the Hamiltonian will be diagonal in an irreducible representation whose basis can be labeled by the complete set of quantum numbers of a particular subgroup chain, and that the energy eigenstates are thus just the basis vectors of this irreducible representation. If the Hamiltonian cannot be expressed solely as a function of the Casimir operators of a single subgroup chain, we say that the dynamical symmetry of the system is broken.

To connect the properties of quantum systems to manifestly classical concepts such as integrability and chaos, the quantum dynamics must be formulated, in a way such that a classical-like limit exists. Almost all physical properties of complex quantum systems are interpreted in terms of the various mean-field theories. Thus, the mean-field approximation is, of necessity, of fundamental importance to our understanding of realistic quantum systems. Therefore, in this Letter, we shall investigate the mean-field motion of quantum systems with dynamical groups to establish the connection between dynamical symmetry breaking and nonintegrability.

An elegant way of formulating a mean-field theory is in terms of the coherent-state path integral. Stationary phase evaluation of the exact propagator in the coherent-state representation leads to approximate equations of motion for the coherent-state parameters. This space can be given a symplectic structure,[6] and the equations are thus the Hamilton's equations for the mean-field motion. For coherent states of a general Lie group G, this space is the coset space G/H, where H is a stability subgroup of G. Thus, the coset space is the "quantum phase space" for the mean-field motion. We have shown that the dimension of a quantum phase space can be uniquely defined from the dynamical group of the system[7]: the dimension of the quantum phase space is always exactly twice the number of quantum numbers of a particular canonical subgroup chain required to label a basis state of a given irreducible representation of G. If the usual classical limit of the system exists,[8] it can be shown that the dimension of the quantum phase space is identical to that of the classical phase space.[7]

Based on this definition of the quantum phase space, an explicit connection between the dynamical symmetry and the integrability can be made: Suppose that, for a quantum system with dynamical group G, the dimension of the complete set of quantum numbers necessary to fully label its Hilbert space (an irreducible carrier space of G) is M. The dimension of the corresponding quantum phase space is then $2M$. When the system possesses dynamical symmetry, there are M good quantum numbers which correspond to M constants of motion of the mean-field equations. The mean-field motion is therefore integrable in the classical mechanical sense: Trajectories are quasiperiodic on invariant M-tori in the quantum phase space. No chaos can exist if the dynamical symmetry is intact.

If the dynamical symmetry of the system is broken, some of the M good quantum numbers are lost. There are then fewer constants of the motion ($<M$). In this case, the mean-field dynamics is nonintegrable, and thus regions of chaotic motion will in general be found.[2,3] We call this chaotic motion *semiclassical chaos,* as it is a quantum characteristic of the system *provided* that the mean-field or semiclassical approximations are valid. In the classical limit of the mean-field theory, it can be shown that all the conclusions given above are identical to those of classical mechanics.[7] This will guarantee that our basic ideas and definitions are self-consistent.

In order to illustrate these general ideas, we consider the simple example of two coupled spins.[9,10] The properties of this system have previously been studied in relation to the problem of quantum chaos. The model Hamiltonian is written as follows:

$$H=(1-\alpha)[J_{1z}+J_{2z}]+\alpha J_{1x}J_{2x}\,, \tag{3}$$

where $0\le\alpha\le 1$ is a coupling constant. The dynamical group of (3) is $SU^1(2)\otimes SU^2(2)$ for spins 1 and 2. There are two dynamical symmetry chains:

$$SU^1(2)\otimes SU^2(2)\supset\begin{cases}SO^1(2)\otimes SO^2(2)\,, & \text{(4a)}\\ SU^{1+2}(2)\supset SO^{1+2}(2)\,. & \text{(4b)}\end{cases}$$

The bases of Hilbert space which carries the irreducible representation (j_1j_2) of $SU^1(2)\otimes SU^2(2)$ are $\{|(j_1j_2)m_1m_2\rangle;\ m_1=j_1,\dots,-j_1,\ m_2=j_2,\dots,-j_2\}$ and $\{|(j_1j_2)jm\rangle;\ j=j_1+j_2,\dots,|j_1-j_2|,\ m=j,\dots,-j\}$ for dynamical symmetry chains (4a) and (4b), respectively. The dynamical symmetries of H can be classified as follows: (a) for $\alpha=0$, H has both dynamical symmetry chains (4a) and (4b); (b) for $\alpha=1$, the system has dynamical symmetry (4a), with the SO(2) axis redefined to be the x axis; and (c) for $0<\alpha<1$, the system's dynamical symmetry is broken.

The mean-field motion of the coupled spin system can be described by use of the coherent states of $SU^1(2)\otimes SU^2(2)$,[11]

$$|p,q\rangle=\exp\left[\sum_{i=1}^{2}(\zeta_iJ_{i+}-\zeta_i^*J_{i-})\,|(j_1j_2)-j_1-j_2\rangle\right]. \tag{5}$$

Correspondingly, the symplectic coordinates of the coherent states are

$$\begin{aligned}2^{-1/2}(q_i+ip_i)&=(2j_i)^{1/2}(\zeta_i/|\zeta_i|)\sin|\zeta_i|\\&=(2j_i)^{1/2}\sin(\tfrac{1}{2}\theta_i)e^{-i\varphi_i}\,,\end{aligned}$$

where (θ_i,φ_i) are spherical coordinates of the coset representative of $SU^1(2)\otimes SU^2(2)/U^1(1)\otimes U^2(1)$ which is isomorphic to $S^2\otimes S^2$. Then the quantum phase space is a four-dimensional compact space $p_i^2+q_i^2\le 4j_i$, as the number of quantum numbers (m_1m_2) or (jm) for the subgroup chains $SO^1(2)\otimes SO^2(2)$ or $SO^{1+2}(2)$ of $SU^1(2)\otimes SU^2(2)$ is 2. The mean-field dynamical equations in the symplectic phase space have the standard canonical form[6]:

$$\begin{aligned}H(p,q)&=\langle p,q|H|p,q\rangle\\&=(1-\alpha)\{\tfrac{1}{2}(p_1^2+q_1^2)+\tfrac{1}{2}(p_2^2+q_2^2)-(j_1+j_2)\}+\tfrac{1}{4}\alpha q_1q_2(4j_1-p_1^2-q_1^2)^{1/2}(4j_2-p_2^2-q_2^2)^{1/2}\,,\end{aligned} \tag{6a}$$

$$dq_i/dt=\partial H(p,q)/\partial p_i,\quad dp_i/dt=-\partial H(p,q)/\partial q_i\,, \tag{6b}$$

where $i=1$ and 2 for spin 1 and spin 2, respectively.

The character of the mean-field solutions can be investigated by numerical trajectory integration and visualized with use of Poincaré surfaces of section.[2,3] (In all calculations, we set $j_1=j_2=1$ and $E=0.01$.)

For $\alpha=0$, the motion is integrable since $\{\vartheta_{iz},H(p,q)\}=0$, where $\vartheta_{ik}=\langle p,q|J_{ik}|p,q\rangle$ and

$$\{f,g\}=\sum_{i=1}^{2}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right)$$

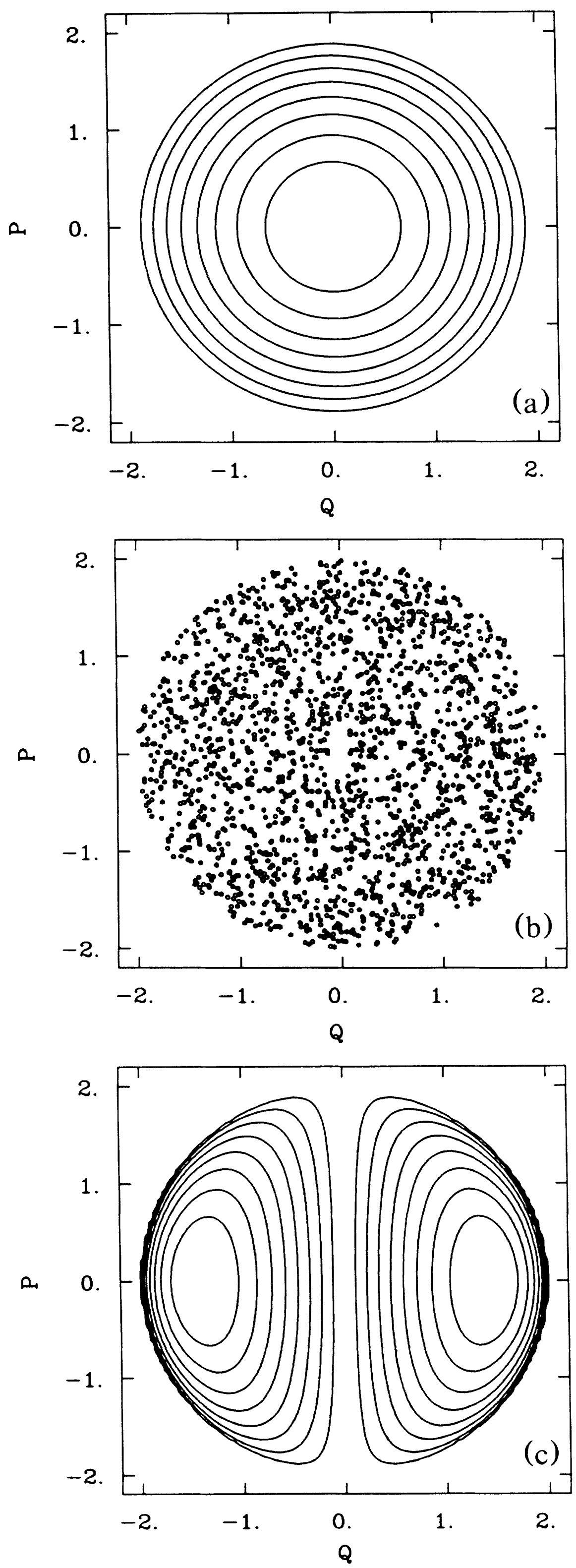

FIG. 1. Surfaces of section for mean-field motion of the coupled spin system (3), in the symplectic coordinates $(q,p)=(q_2,p_2)$ with $j_1=j_2=1$, $E=0.01$. (a) $\alpha=0$; (b) $\alpha=0.8$; (c) $\alpha=1$.

is the Poisson bracket defined on the symplectic structure of the quantum phase space. In the symplectic phase space (p,q), the Hamiltonian is equivalent to two uncoupled harmonic oscillators. The surface of section is shown in Fig. 1(a). This results because there are two inequivalent dynamical symmetries in this limit. In the other limit, $\alpha=1$, the mean-field motion is again integrable, as ϑ_{ix} $(i=1,2)$ are constants of the motion. Figure 1(c) shows the surface of section of this limit. The motion again lies on invariant curves on the surfaces of section.

If $0<\alpha<1$, the system's dynamical symmetries are broken, Eq. (6a) corresponds to a *coupled* harmonic-oscillator system. Because there are no constants of the motion other than the energy E, the mean-field equations are nonintegrable; this manifests itself in the quantum phase space as chaotic trajectories. Figure 1(b) shows this situation, with $\alpha=0.8$. The invariant tori appear to be almost completely destroyed, and the chaotic motion fills the entire surface of the section.

To summarize, for the simple example of a two coupled spin system, we have seen that broken dynamical symmetry leads to nonintegrability of the mean-field motion. As is generic for classical nonintegrable systems, chaotic solutions to Hamilton's ordinary differential equations result from the nonintegrability. For this system, the mean-field motion is identical to classical mechanics (except for the replacement of integer or half-integer j by the continuous j), and the chaotic mean-field motion simply corresponds to chaos in the corresponding classical system. The definition of quantum phase space made here, however, is considerably more general, encompassing systems with no classical analog, such as the many-fermion system in which the Pauli principle plays a very important role.[12] A more detailed and comprehensive discussion will be given in a separate publication.[7]

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