Resolution of Causality Violation in the Classical Radiation Reaction

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We show that the usual derivations of the Lorentz-Dirac equation are not valid at points where the charge's path is a nonanalytic function of time, and that such points inevitably appear when the applied force is freely alterable in the future, i.e, when one considers questions of causality. This enables us to avoid causality violation in classical radiation reaction, making the usual hedge of quantum effects at small distances quite unnecessary.

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It is widely believed that causality violation occurs in the classical theory of point charges,[1-3] and recently its quantum-field-theory analog has been claimed to exist in the Compton scattering of light by protons.[4] In this Letter we show that, for the classical case, causality violation has appeared in previous treatments because of the incorrect use of Taylor expansions for the position of the charge as a function of time.

Taylor expansions are always used to derive the classical equation of motion of a point charge, i.e., the Lorentz-Dirac equation.[3] One usually considers four-momentum conservation for some narrow (radius ϵ) world tube enclosing at least part of the charge's world line. To calculate the flow of electromagnetic four-momentum through the surface Σ of the world tube, one needs to know the current density for the section of the world line enclosed by Σ. So, in order to arrive at a local differential equation of motion for the charge, the usual procedure is to expand the path of the charge $x^a(s)$ in a Taylor series about some particular point P, say (where s is proper time). This then leads to the standard Lorentz-Dirac equation at the point P,

$$ma^a = F^a + \tfrac{2}{3} e^2(\dot{a}^a - a^2 v^a), \tag{1}$$

or simply

$$\ddot{x} - \tau\dddot{x} = f \tag{2}$$

for nonrelativistic motion in one dimension ($f = F/m$, $\tau = 2e^2/3mc^3$, $e^2 = q^2/4\pi\epsilon_0$, the metric is $-+++$, and $c=1$).

We make some remarks on (2). Its general solution is

$$\ddot{x}(t) = e^{(t-b)/\tau}\left[\ddot{x}(b) - \frac{e^{b/\tau}}{\tau}\int_b^t e^{-t'/\tau} f(t')dt'\right]$$

for b arbitrary. In general, this is a runaway solution ($\ddot{x}$ unbounded for large t). Many of the runaway solutions have the additional fault that the charge accelerates in a direction opposite to the applied force. This happens, for example, if one puts $\ddot{x}(b) = 0$. We follow the usual procedure of requiring the absence of runaway solutions as an extra condition, i.e., we impose

$$\ddot{x}(b) = \frac{e^{b/\tau}}{\tau}\int_b^\infty e^{-t'/\tau} f(t')dt',$$

making $\ddot{x}$ finite at infinity. Thus we get

$$\ddot{x}(t) = \frac{e^{t/\tau}}{\tau}\int_t^\infty e^{-t'/\tau} f(t')dt', \tag{3}$$

the usual so-called "preaccelerated" (causality violating) solution,[1-3] for which the acceleration at time t depends on the applied force in the future. Since the casuality violation occurs on an extremely small time scale ($\tau \approx 10^{-23}$ s for an electron), (3) is generally considered to be the most reasonable solution.[1-3]

We note that to derive the Lorentz-Dirac equation at a proper time s, we must assume that the particle's path $x^a(s)$ is an analytic function near time s. (Throughout this paper, "analytic" refers to the real line only, and simply means that a Taylor expansion is valid.) All valid derivations known to the author make use of this assumption. There are derivations which need to assume analyticity over the whole world line, e.g., Dirac's[5] advanced and retarded field method, which considers a world tube enclosing the whole world line. However, there are other derivations which only need to assume analyticity in some neighborhood of the point in question, the size of this neighborhood tending to zero.[6] It is evident that if, for some reason, the path of the charge were nonanalytic at certain points in time, then the usual derivations of the Lorentz-Dirac equation would no longer be valid at such points.

Thus we have the result: *The usual derivations of the Lorentz-Dirac equation (2) are only valid at times such that $x(t)$ and $f(t)$ are analytic functions.* Now the crucial point is that, *when discussing questions of causality, nonanalytic points necessarily arise.* This is because, in such discussions, we must assume that we are free to modify the future externally applied force as we please (e.g., we typically ask: "If we apply a different force for times $t > t_0$, is the acceleration of the charge affected for times $t < t_0$?"). But then the applied force cannot be an analytic function, in general, because knowledge of such

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a function on an interval automatically fixes the function everywhere else. Thus at points in time where one is free to "change one's mind" about what external force to apply to the charge, the external force, and hence also the position of the charge, must, in general, be nonanalytic functions of time. [If $x(t)$ were analytic at such points, the resulting Lorentz-Dirac equation would clearly contradict the nonanalyticity of $f(t)$.] Thus we cannot use Taylor expansions through such points. Using this crucial fact, we can avoid causality violation.

We proceed as follows: At analytic points the Lorentz-Dirac equation may be derived and then used as a condition on the motion of the charge. But at nonanalytic points no such condition may be derived. Since we, so far, have no condition on the motion at nonanalytic points, we are free to impose the condition of causality, which follows from the underlying causal Maxwell theory. This leads consistently to a causal bounded solution.

Consider the following example. A point charge is acted upon by an external force (per unit mass) $f_1(t)$ for times $t<t_0$ and a force $f_2(t)$ for $t>t_0$. We take $f_1(t)$ and $f_2(t)$ to be distinct analytic functions of time. This corresponds to a situation where, at $t=t_0$, one is free to change one's mind about what force to apply to the charge [e.g., $f_1(t)=0$, $f_2(t)\neq 0$ could correspond to a charge at rest which is then pushed at some arbitrary time t_0. Evidently t_0 is a nonanalytic point.] It is natural to assume that, like the applied force, the path of the charge is analytic at all times except $t=t_0$. Thus in this case the Lorentz-Dirac equation (2) is known to be valid for all $t\neq t_0$, but we are given no such condition at $t=t_0$.

We now solve for the motion in this case. First, given the analytic function $f_1(t)$ defined only for $t<t_0$, let $f_{1*}(t)$ be the analytic extension of $f_1(t)$ to all times t. Now let us imagine that $f_{1*}(t)$ were actually applied to the charge for all t. This would correspond to our actual physical situation for $t<t_0$ but not for $t>t_0$. Equation (2) with $f(t)$ replaced by $f_{1*}(t)$ would then be valid everywhere, and we would follow the usual procedure outlined above to arrive at

$$\ddot{x}(t)=\frac{e^{t/\tau}}{\tau}\int_t^\infty e^{-t'/\tau}f_{1*}(t')dt', \tag{4}$$

the usual supposedly causality-violating solution. We stress that (4) would only be valid if the analytic force $f_{1*}(t)$ were actually applied for all time. In that case we would have no causality violation since we would no longer be free to "test causality" by altering the applied force in the future. So far so good. Now (4) was derived by imagining $f_{1*}(t)$ to be applied for all time, but in the real situation we wish to consider, it is actually applied only up to $t=t_0$. However, we can still use (4) to obtain the actual acceleration for $t<t_0$ by using the principle of causality: We can simply say that $\ddot{x}$ for $t<t_0$ is unaffected by what goes on at $t>t_0$, and so (4) must give the actual acceleration for $t<t_0$. For $t>t_0$, the analytic force $f_2(t)$ is applied, and we again find the usual solution

$$\ddot{x}(t)=\frac{e^{t/\tau}}{\tau}\int_t^\infty e^{-t'/\tau}f_2(t')dt'. \tag{5}$$

So the actual acceleration is given by (4) for $t<t_0$ and (5) for $t>t_0$. Since t_0 is the only point at which we are free to alter the applied force, we see that our solution is perfectly causal.

More generally, we argue as follows: Say we have an applied force $f(t)$ which is analytic for times in (a,b), and we wish to find the acceleration $\ddot{x}$ in the interval (a,b). To do this we first analytically extend $f(t)$ from (a,b) to the whole real line, and call the result $f_*(t)$. Then, imagining $f_*(t)$ to act for all times $t>a$ gives us [since the Lorentz-Dirac equation (2) would then be valid for all $t>a$]

$$\ddot{x}(t)=\frac{e^{t/\tau}}{\tau}\int_t^\infty e^{-t'/\tau}f_*(t')dt'. \tag{6}$$

Since, in (a,b), the actual applied force is equal to $f_*(t)$, causality implies that (6) is the actual acceleration for times in (a,b). By assumption of analyticity, we are not free to alter the force in (a,b), and so (6) indeed gives us a causal nonrunaway solution in (a,b).

Hence our general result (6) states: *the acceleration depends not on the force which is actually applied in the future (which would violate causality), but on the analytic extension of the "present" force into the future.*

We note in passing that $\ddot{x}$ jumps at nonanalytic points by an amount

$$\ddot{x}\,|_-^+=\tau^n f^{(n)}\,|_-^+,$$

if $f^{(n)}(t)$ is discontinuous with all other derivatives [and also $f(t)$] continuous. [While the $f^{(m)}(t)$ for $m>n$ are undefined at such a point, we may choose a case where they are otherwise continuous, so that these removable point singularities of $f^{(m)}(t)$ do not contribute to $\ddot{x}\,|_-^+$.] This is perhaps no more peculiar than the usual Newtonian jump in $\ddot{x}$ for a discontinuous force (same as our case $n=0$), noting that our jump in $\ddot{x}$ decreases rapidly as n increases ($\tau\approx 10^{-23}$ s for an electron). However, it makes it evident that the Lorentz-Dirac equation is undefined at nonanalytic points, since $\ddot{x}$ and $\dddot{x}$ are undefined at such points.

We now give two examples of our result (6) in action. First, consider a charge at rest at the origin for all $t<0$, with an external force

$$f(t)=\begin{cases}0 & \text{if } t<0,\\ \lambda(t)\neq 0 & \text{if } t>0.\end{cases}$$

We take $\lambda(t)$ to be analytic so that $t=0$ is the only point at which we are free to choose the applied force. Instead of the usual "preaccelerated" result

$$\ddot{x}(t<0)=\frac{e^{t/\tau}}{\tau}\int_0^\infty e^{-t'/\tau}\lambda(t')dt',$$

we obtain

$$\ddot{x}(t<0)=0,$$

$$\ddot{x}(t>0)=\frac{e^{t/\tau}}{\tau}\int_t^\infty e^{-t'/\tau}\lambda(t')dt'.$$

By the analyticity assumption, $\lambda(t)$ cannot be changed after $t=0$, so everything is causal.

As a second example, we use our result to eliminate a problem raised twelve years ago by Baylis and Huschilt.[7] In the situation they examine, we have an electric field which is constant in some bounded region and zero outside it. They find that, according to the Lorentz-Dirac equation, if a point charge is placed close enough to the edge of this region, then as well as simply remaining at rest, the particle can also preaccelerate into the field region, giving a different possible physical (nonrunaway) solution. In contrast, our general result (6) leads uniquely to the "at rest" solution. Thus, resolution of causality violation seems to simultaneously solve the problem of multiple (nonunique) physical solutions.

As a final point, we note that the relativistic generalization of our result (6) is evidently

$$\ddot{x}^\alpha(s)=\frac{e^{s/\tau}}{\tau}\int_s^\infty e^{-s'/\tau}[f_*^\alpha(s')-\tau\ddot{x}_*^2(s')\dot{x}_*^\alpha(s')]ds', \quad (7)$$

where $x_*^\alpha(s')$ and $f_*^\alpha(s')$ are the analytic extensions of $x^\alpha(s)$ and $f^\alpha(s)$ from a neighborhood of s to all proper times s'. This gives us a causal nonrunaway solution for the relativistic case, the Lorentz-Dirac equation (1) being defined only at proper times such that $x^\alpha(s)$ and $F^\alpha(s)$ are analytic.

We conclude: Since all derivations of the Lorentz-Dirac equation break down at nonanalytic points, we are free to impose causality to propagate the solution through such points. This leads, in a consistent manner, to a general solution which is both bounded and causal. It follows that the usual hedge of quantum effects at small distances is quite unnecessary. In a forthcoming paper, we examine extended charges in the light of the above considerations.[8]

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