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Quantum Distinction of Regular and Chaotic Dissipative Motion

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We generalize the concept of level spacings to dissipative quantum maps. For periodically kicked tops with damping, we find linear and cubic level repulsions under conditions of classically regular and chaotic motion, respectively. The numerically obtained spacing distribution for the chaotic top appears to be universal: It compares favorably with the spacing distribution of general complex matrices of large dimension, the analytical form of which we also present.

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The classical distinction between regular and chaotic motion, based on the concept of Lyapunov exponents, is well known to become meaningless when quantum effects are important. For Hamiltonian systems, however, several intrinsically quantum-mechanical criteria have been established which allow us to distinguish between two types of quantum motion, one going regular and the other chaotic as $\hbar \to 0$. Some of these quantum criteria refer to the spectrum of energy or quasienergy levels (level clustering versus level repulsion[1]), others refer to the temporal behavior of expectation values of typical observables (nearly regular versus erratic quasiperiodicity[2]), and yet others refer to the structure of energy or quasienergy eigenfunctions. Such genuine quantum distinctions are complementary to the classical ones inasmuch as they tend to loose their meaning as $\hbar \to 0$.

For dissipative systems, on the other hand, exclusively quantum-mechanical differences between what becomes regular and chaotic classically are much more elusive than for Hamiltonian ones. The classical difference between, say, a complicated limit cycle and a strange attractor is visible when phase-space structures are considered over several orders of magnitude of action scales. The density matrix (or one of its representatives like, e.g., the Wigner function) reflects such structures on action scales upward of Planck's constant and may thus tell the difference between a strange and a simple attractor with reasonable certainty. But for the density matrix ρ to reveal that the difference in terms of concepts exclusive with quantum mechanics it would have to embody coherences $\langle\psi|\rho|\phi\rangle$ with respect to states $|\psi\rangle$ and $|\phi\rangle$ distinct on action scales large compared to $\hbar$. In the presence of even weak damping, such coherences tend to decay so rapidly that their observation becomes difficult if not impossible. In fact, when the damping rate for probabilities like $\langle\psi|\rho|\psi\rangle$ and $\langle\phi|\rho|\phi\rangle$ is Γ, the decay constant of the coherence $\langle\psi|\rho|\phi\rangle$ tends to be larger by a factor of the order of the action scales inherent in $|\psi\rangle$ and $|\phi\rangle$.

The phenomenon in question is of rather general character. Roughly speaking, dissipative quantum systems tend to go classical more easily than Hamiltonian ones. An illustration of relevance to ongoing research are the difficulty and the impossibility of observing quantum tunneling between states which are, respectively, mesoscopically and macroscopically distinct.[3] The effectively instantaneous decay of coherences between macroscopically distinct states of measuring devices is another illustration important for the quantum theory of measurement[4]; the drastic effects of weak dissipation on recurrence events, both of the regular and the erratic variety, in quantum systems with discrete spectra have also been noted previously.[5]

The foregoing remarks are not meant to imply that the semiclassical behavior of classically chaotic systems with damping is not interesting. Quite the contrary, the gradual unfolding of a strange attractor as $\hbar$ is decreased is an impressive phenomenon quite different from the approach to classically chaotic behavior in Hamiltonian systems.[6] It is well to be aware of an important obstacle, though, when setting out to search for genuine quantum signatures of two types of dissipative quantum dynamics complementary to the well-known classical difference between simple and strange attractors.

We now discuss a quantum distinction found in the numerical treatment of a damped kicked top capable of both dominantly regular and globally chaotic behaviors in the classical limit. The criterion in question is based on the generalization of quasienergies ϕ to complex "levels" $\phi = \phi' + i\phi''$. To appreciate the meaning of these complex ϕ it is crucial to realize that dissipative dynamics must be described in terms of a density matrix. Consequently, for an N-dimensional Hilbert space there are N^2 such ϕ.

In the zero-damping limit the then real ϕ do not reduce to quasienergies, i.e., to eigenphases of the then unitary Floquet operator. Rather, the ϕ become the difference of two quasienergies each N of which vanish while the remaining ones fall into $N(N-1)/2$ pairs $\pm|\phi|$. Only N of these pairs refer to adjacent quasienergy levels, i.e., give nearest-neighbor spacings.

For the weakest of dampings, each ϕ' can still be associated with a pair of quasienergy levels and the corresponding imaginary part ϕ'' with a width. As long as the N nearest-neighbor spacings ϕ' are larger than the width ϕ'' there is no noticeable deviation of the spacing distribution from the zero-damping form. Especially, the distinction between clustering and repulsion of quasienergies is still possible.

However, when the damping is increased such that the width ϕ'' exceeds the zero-damping spacing, the original concept of neighboring levels breaks down. A possible extension of the notion of a spacing is the Euclidean distance between the complex levels in the complex plane. We have found the distribution of smallest distances to display *linear and cubic repulsions under the conditions of classically regular and chaotic motion,* respectively. Linear repulsion, incidentally, is a rigorous property of the Poissonian random processes in the plane. Cubic repulsion, on the other hand, is typical of the non-Hermitian random matrices.

For the system we have investigated the density operator $\rho(t)$ obeys a master equation of the form

$$\dot{\rho}(t) = \Lambda\rho(t) - i[H(t),\rho(t)] \equiv [\Lambda - iL(t)]\rho(t)\,, \quad (1)$$

where Λ is a generator of infinitesimal time translation by damping and $H(t)$ is a Hamiltonian periodically modulated in time by impulsive kicks,

$$H(t) = H_0 + H_i \sum_{n=-\infty}^{+\infty} \delta(t-n)\,. \quad (2)$$

A kick-to-kick stroboscopic description,

$$\rho_{n+1} = e^{\Lambda - iL_0} e^{-iL_1} \rho_n \equiv D\rho_n\,, \quad (3)$$

is therefore convenient with the index n a discrete time counting the number of kicks passed. The complex levels ϕ discussed above are the logarithms of the eigenvalues of the dissipative quantum maps (3), defined by $D\rho = e^{-i\phi}\rho$ where ρ denotes an "eigenvector." In an N-dimensional Hilbert space, the eigenvectors ρ as well as the density matrices ρ_n are N by N while the discrete-time propagator D is an $N \times N \times N \times N$ tetrad. Important restrictions on D are the conservation of (i) probability ($\mathrm{tr}D\rho_n = \mathrm{tr}\rho_n$), (ii) Hermiticity ($\rho_n = \rho_n^+$), and (iii) positivity ($\rho_n > 0$). It follows that each level ϕ is accompanied by its negative complex conjugate $-\phi^*$ as another one. The eigenvalue unity, i.e., $\phi = 0$, arises for a stationary solution, $D\bar{\rho} = \bar{\rho}$. We are interested in stable maps only for which all eigenvalues are smaller than or at most equal to unity in modulus. In a complex plane, the N^2 eigenvalues thus live on a circular disk of unit radius, centered at the origin. Confined to the boundary of that disk in the nondissipative limit they wander inwards as the damping is increased (see Fig. 1); only exceptional ones like the one pertaining to the stationary solution of (3) remain on the circumference.

More specifically, we have worked with kicked spins $\mathbf{J} = (J_x, J_y, J_z)$ of conserved square, $\mathbf{J}^2 = j(j+1)$, $j \gg 1$. The dimension of the Hilbert space is thus $N = 2j+1$. The Hamiltonian part of the dynamics was chosen similarly as in previous work[2,5,7]

$$H_0 = pJ_z + k_0 J_z^2/2j, \quad H_1 = k_1 J_y^2/2j\,, \quad (4)$$

with coupling constants p, k_0, and k_1. The damping gen-

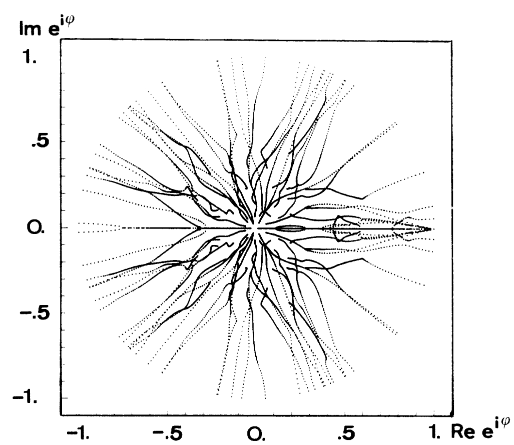

FIG. 1. Position of the complex eigenvalues of the propagator D [Eq. (3)] ($j=6$, $p=2$, $k_1=8$, $k_0=10$, even parity) for different values of the damping constant Γ. ($\Gamma=0$ to 0.4 in steps of $\Delta\Gamma=0.005$.)

erator Λ was chosen as

$$\Lambda\rho = (\Gamma/2j)\{[J_-,\rho J_+] + [J_-\rho, J_+]\}\,, \tag{5}$$

with $J_\pm = J_x \pm iJ_y$ as is well known from the theory of superfluorescence.[8] Taken by itself, Λ would describe the relaxation of $\rho(t)$ towards the state of minimal J_z on the time scale $1/\Gamma$. The dynamics [(3), (4), and (5)] correspond to a classically nonintegrable motion if $k_1 \neq 0$. For all coupling constants p, k_0, k_1, and Γ smaller than unity, however, the classical sphere $\mathbf{J}^2 = \text{const}$ is dominated by regular motion. For sufficiently strong nonlinearities, there is global chaos corresponding, for $\Gamma \neq 0$, to a strange attractor. We shall now be concerned with a quantum analog of the classical transition from regular to chaotic motion when k_1 is increased from zero to sufficiently large values.

Our results for the eigenvalues of D are illustrated in Figs. 1 and 2. The diagonalization of D was carried out for $j=10$ so that D was $21\times21\times21\times21$. (Actually, because of a parity D breaks up into two blocks of about equal size.) Figure 1, as already mentioned, displays the inward migration of the N^2 complex levels from the circumference of the unit circle with increasing damping. In order to obtain meaningful spacing distributions for fixed coupling constants, the coordinate mesh within the unit circle was rescaled so as to obtain a uniform distribution of points $\exp(-i\phi)$. Figure 2 depicts, under the label "regular," the integrated spacing distribution for the classically integrable case $k_1=0$; the damping constant was set to $\Gamma=0.1$ and the linear precession parameter to $p=2$; to smooth the staircase, 100 different values of k_0 between 10 and 12 were chosen and the corresponding distributions thrown together. The behavior shown here for $k_1=0$ actually prevails for nonzero

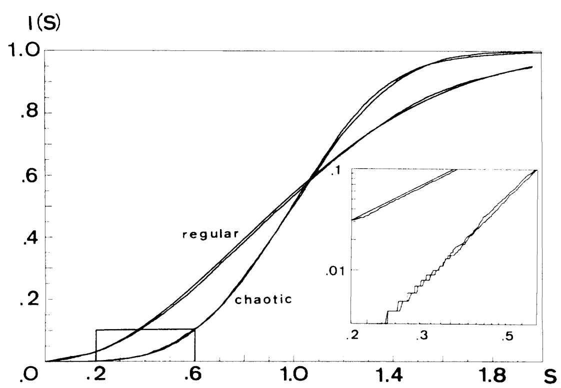

FIG. 2. Integrated spacing distributions $I(S)$ for two different universality classes. In each of the two pairs one curve refers to our map D with $j=10$, $\Gamma=0.1$, and $p=2$, and 100 different values of k_0 between 10 and 12; the "regular" case pertains to $k_1=0$ and the "chaotic" one to $k_1=8$. The second line in the "regular" pair represents the integrated Wigner distribution (6) and in the "chaotic" pair the spacing staircase of general non-Hermitian matrices of like dimension.

values of k_1, $k_1 \lesssim 0.2$; in that range of k_1 all classical trajectories quickly approach a fixed point. The second line in the "regular" pair represents the asymptotic $(N\to\infty)$ result for a Poissonian random process in the plane,

$$P_P(S) = \tfrac{1}{2}\pi S\, e^{-\pi S^2/4}. \tag{6}$$

Interestingly, the so-called Wigner distribution (6) was previously known to hold rigorously for the Gaussian orthogonal ensemble of real symmetric (2×2) matrices; here it arises in a different context as rigorous in the limit $N\to\infty$. At any rate, linear repulsion of the complex levels in this integrable case is obvious and the agreement between the results for the Poissonian random process and the map D is quite satisfactory.

One of the two curves labeled "chaotic" in Fig. 2 was obtained under conditions identical to those of the integrable case, except that $k_1=8$ was chosen to make the classical motion strongly chaotic. The other line in that pair corresponds to a general complex random matrix of similar dimension with Gaussian statistics. Cubic repulsion can be inferred as well as impressive agreement in the behaviors of the map D and the random matrix. The latter agreement might at first sight be astonishing since the general random matrix does not respect any of the restrictions the map D must obey; in the limit of large dimension the latter restrictions seem not to affect the local fluctuations in the complex spectrum.

Figure 2 suggests that general complex random matrices are of relevance for the chaotic case. We have therefore taken Ginibre's[9,10] joint probability of the eigenvalues of such matrices and calculated the corresponding spacing distribution in the limit $N\to\infty$. We choose units such that the N eigenvalues tend to cover, with uniform mean density, a circle around the origin of radius $\sqrt{N}$ in the complex plane.[9,11] We define $p(s)ds$ as the probability that the distance of a randomly chosen eigenvalue to its nearest neighbor lies in the interval $(s, s+ds)$ and derive $p(s)$ from its integral as $p(s)=di(s)/ds$. In fact, it can be shown that $1-i(s)$, i.e., the probability that a randomly chosen eigenvalue has all its neighbors further away than s, takes the form of a product,

$$1-i(s) = \lim_{N\to\infty}\prod_{n=1}^{N-1}[e_n(s^2)e^{-s^2}]\,,$$
$$e_n(x) = 1 + x/1! + x^2/2! + \cdots + x^n/n!. \tag{7}$$

Interestingly, if the limit $N\to\infty$ in (7) is not taken, the finite product is the conditional probability that if one eigenvalue lies *at the origin,* all $N-1$ others are further away than s. It is only for $N\to\infty$ that full homogeneity arises within the circle in question and that $p(s)$ becomes independent of the reference point.

We were pleased to realize that the infinite product in (7) converges quite rapidly and is easy to compute. Im-

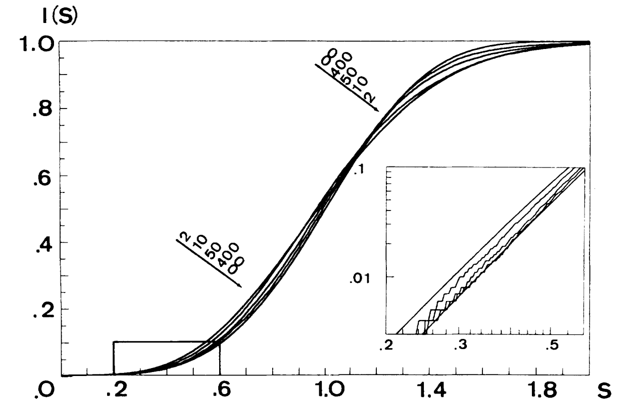

FIG. 3. The integrated spacing distribution for general complex random matrices for matrices with $N=2$ [see (10)], 10, 50, 400, and ∞ [see (7)]. Inset: Common cubic level repulsion for small spacings with logarithmic axes.

mediately accessible is the behavior for small s,

$$i(s)=\frac{s^4}{2}-\frac{s^6}{6}+\frac{s^8}{24}-\frac{11}{120}s^{10}+O(s^{12}). \tag{8}$$

Somewhat more delicate is the asymptotic large-s behavior. By using the Euler-MacLaurin summation formula,[12] we find

$$-\ln[1-i(s)]=\frac{s^4}{4}+s^2[\ln s+O(1)]. \tag{9}$$

To compare with the numerical data, we have rescaled as $I(S)=i(S\bar{s})$ where $\bar{s}=\int_0^\infty ds[1-i(s)]=1.142929\ldots$, $\bar{S}=1$. Figure 3 shows the numerically obtained spacing distributions for random matrices of dimensions 10, 50, and 400. We also show the integral of the spacing density for $N=2$,

$$P(S)=(3^4\pi^2 2^{-7})S^3 e^{-(3^2\pi 2^{-4})S^2}, \tag{10}$$

as well as the asymptotic result corresponding to Eq. (7). The N dependence of these distributions is rather more strongly pronounced than in the well-known cases of Hermitian and unitary matrices. The reason is a boundary effect. If we discard eigenvalues too close to the circumference of the circle, excellent agreement with the asymptotic result is reached for $N>20$.

We would like to conclude with a question concerning universality classes of dissipative systems: Are there, beyond the ones with linear and cubic repulsions, other classes? To check on the possibility of quadratic repulsion we have set $k_0=0$ in (4) and looked at the underlying Hamiltonian dynamics of the top coupled to a heat bath which yields the damped motion (3) upon elimination of the heat bath. The Hamiltonian dynamics in question has a unitary Floquet operator with, because of $k_0=0$, a time-reversal symmetry[7] and thus linear repulsion of its quasienergies. As we have checked numerically, however, the complex levels of the dissipative map (3) maintain their cubic repulsion even for $k_0=0$. Further investigations for different model systems will be necessary to establish a definitive answer to the question touched upon here. Of special interest are autonomous dissipative systems since these are known to have or not to have detailed balance depending on whether or not the underlying microscopic dynamics has a time-reversal invariance.[13]

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