Electroweak Radiative Corrections to τ Decay

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An analysis of electroweak radiative corrections to τ decays is presented. Precise predictions for leptonic and some hadronic decay rates are given. The total hadronic decay width is shown to be relatively enhanced by 2.36% due mainly to short-distance loop effects. Implications for the extraction of $\Lambda_{\overline{MS}}$ from the τ lifetime or leptonic branching ratios are discussed. ($\overline{MS}$ denotes the modified minimal-subtraction scheme.)

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The standard $SU(3)_c \otimes SU(2)_L \otimes U(1)$ model of strong and electroweak interactions is a renormalizable quantum field theory. Radiative corrections to physical processes are, therefore, finite and calculable within that framework. As such, one can test the standard model at the level of its quantum corrections and search for hints of "new physics" by comparing precise experimental data with theoretical predictions.

In this Letter, we describe the effect of electroweak radiative corrections on τ decays.[1] Our analysis is prompted by (1) recent improvements in measurements of the τ lifetime and branching ratios as well as good prospects for further significant improvements,[2] and (2) an analysis of the QCD corrections to the total hadronic decay width of the τ by Braaten[3] which suggests that QCD perturbation theory is applicable. That being the case, measurements of the τ lifetime or leptonic branching ratios can in principle provide an extremely accurate determination of $\Lambda_{\overline{MS}}$, the QCD mass scale. ($\overline{MS}$ denotes the modified minimal-subtraction scheme.) We comment further on that possibility after an account of electroweak radiative corrections has been given.

We begin by considering the electroweak radiative corrections to the leptonic decays $\tau \to e\bar{\nu}_e\nu_\tau$ and $\tau \to \mu\bar{\nu}_\mu\nu_\tau$. The simplest strategy is to compute residual $O(\alpha)$ loop corrections after normalizing the lowest-order decay-rate prediction in terms of G_μ, the muon decay constant,

$$G_\mu = (1.166\,37 \pm 0.000\,02) \times 10^{-5}\ \mathrm{GeV}^{-2}. \qquad (1)$$

That precisely determined parameter is extracted from the radiatively corrected total decay rate of the muon[4,5]

$$\Gamma(\mu \to \mathrm{all}) = \frac{G_\mu^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e^2}{m_\mu^2}\right)\left(1+\frac{3}{5}\frac{m_\mu^2}{m_W^2}\right)\left[1+\frac{\alpha(m_\mu)}{2\pi}\left(\tfrac{25}{4}-\pi^2\right)\right], \qquad (2a)$$

$$f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x, \qquad (2b)$$

$$\alpha^{-1}(m_\mu) = \alpha^{-1} - \frac{2}{3\pi}\ln\left(\frac{m_\mu}{m_e}\right) + \frac{1}{6\pi} \simeq 136, \qquad (2c)$$

which corresponds to the inclusive sum

$$\Gamma(\mu \to e\nu\bar{\nu}) + \Gamma(\mu \to e\nu\bar{\nu}\gamma) + \Gamma(\mu \to e\nu\bar{\nu}\gamma\gamma) + \Gamma(\mu \to e\nu\bar{\nu}e^+e^-) + \cdots .$$

Note that we have expressed the $O(\alpha)$ electroweak corrections not absorbed in G_μ in terms of an effective $\alpha(m_\mu)$ appropriate for muon-decay energies.[6] In that way, all leading logarithmic corrections of the form $\alpha^{n+1}\ln^n(m_\mu/m_e)$, $n=1,2,\ldots$, are included in (2). The value of G_μ in (1) is obtained by our comparing (2) with the measured muon lifetime[7]

$$\tau_\mu = (2.197\,035 \pm 0.000\,040) \times 10^{-6}\ \mathrm{s}, \qquad (3)$$

with $m_W \simeq 80.9$ GeV and

$$m_\mu = 105.659\,16 \pm 0.000\,30\ \mathrm{MeV}, \qquad (4a)$$

$$m_e = 0.511\,0034 \pm 0.000\,0014\ \mathrm{MeV}. \qquad (4b)$$

In the case of leptonic τ decays, the electroweak radiative corrections are very simple. $SU(2)_L \otimes U(1)$ gauge symmetry implies $\tau - \mu - e$ universality. Hence, there is a one-to-one correspondence between loop corrections to $\mu \to e\bar{\nu}_e \nu_\mu$ and $\tau \to e\bar{\nu}_e \nu_\tau$. Indeed, the (radiative inclusive) decay rate for $\Gamma(\tau \to e\nu\bar{\nu}) + \Gamma(\tau \to e\nu\bar{\nu}\gamma) + \Gamma(\tau \to e\nu\bar{\nu}\gamma\gamma) + \cdots$ (the ellipsis also includes all processes in which the virtual photons annihilate into light-fermion pairs), which we denote by $\Gamma(\tau \to e\nu\bar{\nu}(\gamma))$, is simply obtained from (2) under the replacement $m_\mu \to m_\tau$. (We assume massless neutrinos throughout this paper.) Explicitly, one finds

$$\Gamma(\tau \to e\nu\bar{\nu}(\gamma)) = \frac{G_\mu^2 m_\tau^5}{192\pi^3} f\left(\frac{m_e^2}{m_\tau^2}\right)\left(1+\frac{3}{5}\frac{m_\tau^2}{m_W^2}\right)\left[1+\frac{\alpha(m_\tau)}{2\pi}\left(\tfrac{25}{4}-\pi^2\right)\right], \tag{5}$$

with[8]

$$\alpha^{-1}(m_\tau) \simeq 133.3\,, \tag{6a}$$

$$m_\tau = 1784.2 \pm 3.2\ \text{MeV}\,. \tag{6b}$$

Similarly, for the muon-decay mode,

$$\Gamma(\tau \to \mu\nu\bar{\nu}(\gamma)) = \frac{G_\mu^2 m_\tau^5}{192\pi^3} f\left(\frac{m_\mu^2}{m_\tau^2}\right)\left(1+\frac{3}{5}\frac{m_\tau^2}{m_W^2}\right)\left[1+\frac{\alpha(m_\tau)}{2\pi}\left(\tfrac{25}{4}-\pi^2\right)\right], \tag{7}$$

where we have neglected corrections of order $(\alpha/2\pi)m_\mu^2/m_\tau^2$. Using the values in (1), (4), and (6) leads to the standard-model predictions.

$$\begin{aligned}&\Gamma(\tau \to e\nu\bar{\nu}(\gamma)) = (4.115 \pm 0.037)\times 10^{-13}\ \text{GeV}\,,\\ &\Gamma(\tau \to \mu\nu\bar{\nu}(\gamma)) = 0.9728\,\Gamma(\tau \to e\nu\bar{\nu}(\gamma)) = (4.003 \pm 0.036)\times 10^{-13}\ \text{GeV}\,,\end{aligned} \tag{8}$$

where the 0.9% uncertainty comes entirely from the uncertainty in m_τ. That uncertainty will significantly diminish as m_τ is better determined. Any experimental deviation from the predictions in (8) would indicate the presence of new physics beyond the standard model.[9]

Electroweak radiative corrections to hadronic decays of the τ are more interesting. Normalizing the lowest-order semihadronic amplitudes in terms of G_μ, quantum loop corrections are still finite and calculable. However, because of the fractional electric charge carried by quarks [they have different U(1) quantum numbers from lepton doublets], short-distance loop corrections involving γ and Z bosons distinguish leptonic and semihadronic amplitudes.[5] This distinction has long been known to play an important role in β decay[5] and more recently in deep-inelastic $\nu_\mu N \to \mu^- X$ scattering.[10] Indeed, in those cases the short-distance radiative corrections are crucial for reconciling theory and experiment. Adjusting the general current-algebra results in Ref. 11 to the case of τ decay, we find that semihadronic decay amplitudes for the τ are enhanced by a factor

$$1+\frac{3\alpha}{4\pi}(1+2\bar{Q})\ln\left(\frac{m_Z}{m_\tau}\right), \tag{9}$$

where $m_Z = 91.9$ GeV and $\bar{Q}$ is the average quark-doublet charge $\bar{Q} = \frac{1}{2}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{6}$. The explicit use of $\bar{Q} = \frac{1}{6}$ in (9) gives

$$1+\frac{\alpha}{\pi}\ln\left(\frac{m_Z}{m_\tau}\right) \simeq 1.00916\,. \tag{10}$$

(Note, for leptons $\bar{Q} = -\frac{1}{2}$; so there is no analogous logarithm.) We can go even further and sum up all short-distance logarithms of the form $\alpha^n \ln^n m_Z$ via the renormalization group.[8] That procedure replaces (10) by (in the notation of Ref. 8)

$$S^{1/2}(m_\tau, m_Z) = \left(\frac{\alpha(m_b)}{\alpha(m_\tau)}\right)^{9/38}\left(\frac{\alpha(m_t)}{\alpha(m_b)}\right)^{9/40}\left(\frac{\alpha(m_W)}{\alpha(m_t)}\right)^{3/16}\left(\frac{\alpha(m_Z)}{\alpha(m_W)}\right)^{6/11}, \tag{11}$$

where for $m_t \simeq 45$ GeV

$$\begin{aligned}&\alpha^{-1}(m_Z = 91.9\ \text{GeV}) = 127.71\,,\\ &\alpha^{-1}(m_W = 80.9\ \text{GeV}) = 127.78\,,\\ &\alpha^{-1}(m_t = 45\ \text{GeV}) = 128.78\,,\\ &\alpha^{-1}(m_b = 4.5\ \text{GeV}) = 132.04\,,\\ &\alpha^{-1}(m_\tau = 1.78\ \text{GeV}) = 133.29\,,\end{aligned} \tag{12}$$

and one finds

$$S^{1/2}(m_\tau, m_Z) \simeq 1.00966\,, \tag{13}$$

which is to be compared with the leading $O(\alpha)$ term in (10). One can also calculate QCD corrections[5,8] to $S(m_\tau, m_Z)$. They are small, and reduce (13) by about 0.00011. We therefore employ

$$S(m_\tau, m_Z) = 1.0192 \tag{14}$$

as the short-distance semihadronic-decay-rate enhancement factor. There are, of course, additional low-frequency ($<m_\tau$) $O(\alpha)$ loop corrections that depend in general on hadronic structure. They are not enhanced by logarithms; so, we assign an uncertainty of about $\pm 0.5\%$ to them. To calculate such effects would require a model of hadronic structure.

Having evaluated the leading short-distance enhancement to hadronic decay amplitudes of the τ, we now focus on the total hadronic width. Following Braaten,[3] we consider the quantity

$$R_H \equiv \Gamma(\tau \to \nu_\tau + \text{hadrons})/\Gamma(\tau \to e\nu\bar{\nu}(\gamma)), \quad (15)$$

which is analogous to the ratio $\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ studied in e^+e^- annihilation.[12] In terms of R_H, the total τ decay rate (inverse lifetime) and leptonic branching ratios are given by

$$\Gamma(\tau \to \text{all}) = \tau^{-1} = (1.9728 + R_H)\Gamma(\tau \to e\nu\bar{\nu}(\gamma)), \quad (16)$$

$$B(\tau \to e\nu\bar{\nu}(\gamma)) = B(\tau \to \mu\nu\bar{\nu}(\gamma))/0.9728 = (1.9728 + R_H)^{-1}. \quad (17)$$

So, R_H can be obtained independently from lifetime or branching-ratio measurements. Using the current experimental averages, we find[13]

$$\tau_\tau = (2.98 \pm 0.08) \times 10^{-13}\,\text{s} \to R_H = 3.39 \pm 0.15, \quad (18a)$$

$$B(\tau \to e\nu\bar{\nu}(\gamma)) = 0.177 \pm 0.004 \to R_H = 3.68 \pm 0.13, \quad (18b)$$

$$B(\tau \to \mu\nu\bar{\nu}(\gamma)) = 0.179 \pm 0.004 \to R_H = 3.46 \pm 0.12. \quad (18c)$$

Together, they give a world average

$$R_H^{\text{ave}} = 3.52 \pm 0.08, \quad (19)$$

which is consistent with all the independent determinations in (18). [It suggests that some upward shift in $B(\tau \to e\nu\bar{\nu}(\gamma))$, to about 18.2%–18.4%, is likely.] In the future, we expect a significant reduction of the error in (19) from precise lifetime as well as branching-ratio measurements.

The standard model's theoretical prediction for R_H can be computed with perturbative QCD in conjunction with the electroweak corrections in (5) and (14). Including our previous estimate of the neglected $O(\alpha)$ uncertainty, we find

$$R_H = 3(|V_{ud}|^2 + |V_{us}|^2)(1.0236 \pm 0.0050) \times (1 + \text{QCD corrections}), \quad (20)$$

where V_{ud} and V_{us} are Cabibbo-Kobayashi-Maskawa matrix elements and 3 is a quark color factor. [Note, that the 2.36% electroweak correction corresponds to the combined effect of (14) and the last factor in (5).] The QCD corrections have been estimated by Braaten[3] to $O(\alpha_s^2)$ to be about +10% for an effective three-flavor $\Lambda_{\overline{MS}}^{(3)} = 150$ MeV (corresponding[14] to an effective four-flavor $\Lambda_{\overline{MS}}^{(4)} \simeq 120$ MeV). The recently computed[15] $O(\alpha_s^3)$ correction increases that estimate to about[16] 15% (an uncomfortably large shift). Using a +15% QCD correction in (20) leads to the prediction (for[17] $|V_{ud}|^2 + |V_{us}|^2 \simeq 0.9979$)

$$R_H^{\text{theory}} \simeq 3.52 \ (\text{for } \Lambda_{\overline{MS}}^{(3)} \simeq 150 \text{ MeV}), \quad (21)$$

which is right on the world average in (19). We have not quoted an error in (21) because we believe that theoretical uncertainties in the QCD calculation, particularly the $O(\alpha_s^3)$ effect, need further study. [The experimental uncertainty in (19) would lead to a ± 30-MeV error in the extraction of $\Lambda_{\overline{MS}}$ if there were no theory uncertainty.] Nevertheless, the excellent agreement between theory and experiment for a reasonable (albeit somewhat small) $\Lambda_{\overline{MS}}$ gives us some degree of confidence in the reliability of perturbative QCD for analyzing τ decay. It suggests a strategy for precisely determining $\Lambda_{\overline{MS}}$. Reduce the experimental uncertainty in R_H to a negligible level (say $< \frac{1}{2}\%$). Then use the formula in (20) with the explicit QCD corrections[3,16] to determine $\Lambda_{\overline{MS}}$. If the theoretical uncertainty in (20) can be reliably reduced to $<2\%$, then a determination of $\Lambda_{\overline{MS}}$ to within about ± 25 MeV will be possible; an impressive accomplishment. τ decays would then provide the best experimental determination of that fundamental mass scale and a standard against which other experimental measurements would be compared.

Our calculation of the short-distance electroweak correction to hadronic decays can also be applied to specific exclusive decays.[9,18] For example, consider the decay $\tau \to \nu_\tau + \pi$. In that case, it is best to eliminate the dependence on f_π by comparing with $\pi \to \mu\bar{\nu}_\mu$. Including the leading $O(\alpha \ln m_Z)$ correction to both decays, we find

$$\frac{\Gamma(\tau \to \nu_\tau\pi)}{\Gamma(\pi \to \mu\bar{\nu}_\mu)} = \frac{1}{2}\frac{m_\tau^3}{m_\pi m_\mu^2}\frac{(1 - m_\pi^2/m_\tau^2)^2}{(1 - m_\mu^2/m_\pi^2)^2}\frac{[1 + (2\alpha/\pi)\ln(m_Z/m_\tau)]}{1 + \frac{3}{2}(\alpha/\pi)\ln(m_Z/m_\pi) + \frac{1}{2}(\alpha/\pi)\ln(m_Z/m_\rho)}. \quad (22)$$

(Note that both decays are enhanced by logarithmic electroweak corrections.)

Using $m_\pi = 139.57$ MeV and $m_\rho = 770$ MeV along with

$$\Gamma(\pi \to \mu\bar{\nu}_\mu) = (2.526 \pm 0.002) \times 10^{-17}\,\text{GeV} \quad (23)$$

leads to the predictions[19]

$$\Gamma(\tau \to \nu_\tau \pi) = 2.472\times 10^{-13}\ \mathrm{GeV}\,,$$
$$\Gamma(\tau \to \nu_\tau \pi)/\Gamma(\tau \to e\nu\bar{\nu}(\gamma)) = 0.601\,. \tag{24}$$

A similar analysis for the decay $\tau \to \nu_\tau K$ leads to the predictions

$$\Gamma(\tau \to \nu_\tau K) = 1.64\times 10^{-14}\ \mathrm{GeV}\,,$$
$$\Gamma(\tau \to \nu_\tau K)/\Gamma(\tau \to \nu_\tau \pi) = 0.066\,. \tag{25}$$

In the case of other exclusive hadronic decay rates of the τ estimated relative to $\Gamma(\tau \to e\nu\bar{\nu}(\gamma))$, one should as a first approximation increase previous estimates by 2.36% to account for the electroweak correction. For example, rescaling an estimate of $\Gamma(\tau \to \nu_\tau \pi^0 \pi^-)$ by Gilman and Rhie[18] with our 2.36% enhancement leads to the prediction

$$\frac{\Gamma(\tau \to \nu_\tau \pi^0 \pi^-)}{\Gamma(\tau \to e\nu\bar{\nu}(\gamma))} = 1.26\,. \tag{26}$$

Other hadronic-decay-rate estimates can be similarly adjusted. The resulting enhancements decrease, somewhat, the apparent deficit in single-charged-prong τ decays recently reported.[18,20,21] They do not, however, solve the problem. (Its resolution probably lies in the experimental measurements and their normalization.)

In conclusion, we have presented an analysis of electroweak radiative corrections to τ decays. Our results provide precise predictions for leptonic and some hadronic decay rates, which can be tested as experiments achieve higher accuracy. The 2.36% enhancement of the hadronic decay width can have important consequences. Its inclusion is necessary if one is to extract a precise value of $\Lambda_{\overline{MS}}$ from measurements of R_H. At present, a comparison of theory (to order α_s^3) and the experimental value of R_H is consistent with $\Lambda_{\overline{MS}}^{(3)} \simeq 150$ MeV ($\Lambda_{\overline{MS}}^{(4)} \simeq 120$ MeV). The experimental uncertainty is about ± 30 MeV [from (19)], but theoretical uncertainties require further study.

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