Cascade and Intermittency Model for Turbulent Compressible Self-Gravitating Matter and Self-Binding Phase-Space Density Fluctuations

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A simple physical model which describes the dynamics of turbulence and the spectrum of density fluctuations in compressible, self-gravitating matter and self-binding, phase-space density fluctuations is presented. The two systems are analogous to each other in that each tends to self-organize into hierarchical structures via the mechanism of Jeans collapse. The model, the essential physical ingredient of which is a cascade constrained by the physical requirement of quasivirialization, is shown to exhibit interesting geometric properties such as intrinsic intermittency and anisotropy.

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A series of recent numerical simulations by Berman, Tetreault, and Dupree[1] has provided compelling evidence for the existence and importance of nonlinear, ballistic (i.e., nonwavelike) fluctuations in a turbulent, collisionless plasma. These fluctuations, which have come to be dubbed "clumps,"[2] can take the form of either enhancements or depletions (holes[3,4]) in the phase-space density. In the form of depletions, phase-space density structures exhibit a disposition toward self-binding, while in the form of enhancements, such material self-repels and fills the interstitial region between holes. The simulations in question, which are of an unmagnetized, single-species plasma, track the evolution of an initially periodic "checkerboard" distribution of local excesses and cavities of charged particles in one-dimensional phase space (i) under the influence of their own forces (a situation referred to as *decay turbulence*), and separately, (ii) under the action of an externally imposed electrostatic field spectrum (*forced* or *driven* turbulence). The decay turbulence was observed to evolve along two disparate time scales: On a fast time scale, the initial, ordered arrangement degenerated into a fully turbulent, random distribution of hole sizes ranging from the initial structural dimensions down to the smallest resolution of the simulation. On a longer time scale, coalescence of holes took place, leading asymptotically in time to a spatially intermittent distribution of a few deep (i.e., large amplitude) phase-space holes, with small-amplitude, positive fluctuations filling the interstitial region in between. The probability of finding a fluctuation in a phase-space cell was found to become more skewed, both in time and with decreasing size, in favor of negative fluctuations. In simulations of forced turbulence, the initial structures were similarly observed to fissure into a distribution of different size holes; however, for the forcing function chosen, the probability distribution remained Gaussian. The similarity between these pictures and corresponding simulations in fluid turbulence[5] is striking, and suggests a more profound affinity between the two disciplines than heretofore expected on the basis of the traditional paradigm of a plasma as a system of interacting waves and discrete particles. In this Letter, we propose a simple dynamic model with which one can explain the genesis of holes over a hierarchy of scales, together with their intermittent distribution in phase space, by interpreting the self-trapping tendency of holes in terms of proximity to Jeans marginality. By way of motivation, we shall first apply the model to compressible, self-gravitating matter, thus attempting to describe some of the observational evidence for the velocity-dispersion–density–length-scale correlations and morphology of molecular clouds in the interstellar medium (ISM).

Observations of large density fluctuations and supersonic internal velocity dispersions (inferred from molecular spectral emissions) have confirmed that the ISM is strongly compressible over a wide range of scales [0.1 to 200 pc (pc denotes parsecs)].[6] Moreover, the self-gravitational force which operates between clouds over all scales is believed to play a pivotal role in regulating the dynamics of the ISM. Magnetic fields and stellar winds are potentially important additional ingredients to this overall picture. The cloud complexes observed have an embedded, hierarchical organization and display windswept, filamentary or ribbonlike structure down to the smallest observable scale.[7,8] As first noted by Larson[9] and confirmed by several later studies,[6] the available data manifest a remarkable power-law correlation among the velocity dispersion v, the density fluctuation ρ, and the length scale l: $v \sim l^p$, $0.3 \lesssim p \lesssim 0.6$, and $\rho \sim l^q$, $-1.4 \lesssim q \lesssim -0.75$, spanning three decades of cloud size (0.1 pc $\lesssim l \lesssim 200$ pc), and operative both within an individual cloud as well as in between different clouds. Of equal significance is the observation that the clouds over this same range are in approximate virial balance; i.e., the kinetic energy associated with their linewidths approximately balances one-half their gravitational energy.

Elsewhere we shall present a model which seeks to account for all the various features discussed above. In order to motivate the scenario of proximity to Jeans marginality, however, a simplified version ignoring magnetic fields and stellar-wind sources will suffice here. Consistent with observations of supersonic flows, we envision the ISM as a turbulent, *driven* system, with energy being injected at the largest scales by hydrodynamic sources, such as the shear associated with differential galactic rotation,[10] or the subscale instability of spiral density waves. The energy is then transferred from large to small scales in a two-stage cascade sequences[11]:

(i) Catastrophic or collapse phase.—On a fast time scale, a given cloud, being strongly Jeans supercritical (see discussion below), undergoes gravitational collapse. In the process of the collapse, the change in the gravitational energy is channeled into random or turbulent kinetic energy[6] until a quasivirialized state is achieved where the turbulent stresses thus generated balance the gravitational force, and bring the collapse to a halt.

(ii) Inertial cascade phase.—On a longer time scale, the cloud fragments into smaller subconstituents by the shearing action of tidal forces. These fragments no longer have sufficient random kinetic energy to support against free-fall collapse, and the process repeats itself. This scenario continues onto the smallest scales in the hierarchy (~ 0.1 pc) at which point clouds become subsonic and thermalize into the ambient intercloud gas.

It is conjectured that, superposed on the large-scale flow, supersonic shock motion induces a progressively more localized and patchy distribution of dissipation. We then have a fractally homogeneous turbulence situation, the essence of which is adequately captured by the absolute curdling or β model of Mandelbrot[12] and Frisch, Sulem, and Nelkin.[13] This model modifies the Kolmogorov cascade (fragmentation) sequence to allow the eddies (clouds) at each step n of the hierarchy to fill only a fraction, $\beta=(l_n/l_{n-1})^\mu$, of the preceding generation's volume, assuming that the largest eddies (clouds) were space filling. The exponent appearing in the expression for β, i.e., $\mu=d-D$ (where $d=3$ is the topological or Euclidean dimension and D the self-similarity or fractal dimension), is referred to as the codimension or intermittency exponent and is a measure of the degree to which the dissipative structures fill space ($D=d \Longleftrightarrow$ space filling). Our model differs from that of Ferrini, Marchesoni, and Vulpiani,[11] both in our use of quasivirialization to relate velocity dispersion, density fluctuation, and length scale together in lieu of an *ad hoc* relationship, and also in our interpretation of "active" and "inert" ("frozen" in the Ferrini, Marchesoni, and Vulpiani terminology) flow regions. In contrast to the latter work, we understand active regions to refer to the molecular clouds themselves as opposed to the ambient intercloud gas since, relative mass ratios notwithstanding,[11] it is in the former that the *dynamic transfer of energy from large to small scales* is taking place.

Quasivirialization can be given a useful interpretation in terms of marginality with respect to Jeans instability. The latter expresses the competition between (outward) pressure or stress forces, and (inward) gravitational forces:

$$\gamma^2 \simeq \tau_c^{-2}-(c_s^2+v^2)/l^2, \qquad (1)$$

where γ is the (exponential) growth rate, $\tau_c=(4\pi\rho G)^{-1}$ is the gravitational collapse time, G is the gravitational constant, c_s is the sound speed, and v is the internal turbulent velocity shear of the fluid. The marginal state, for strongly supersonic velocity shears characteristic of the observations ($v \gg c_s$), corresponds precisely to what we mean by quasivirialization, i.e., $v \sim l/\tau_c \gg c_s$. Modeling the ISM as an intermittent (i.e., non-space-filling) hierarchy of quasivirialized clouds, we can identify four relevant time scales in the problem: (i) the *local* shearing time $\tau_n^s=l_n/v_n$ (where the subscript n refers to the hierarchy level), (ii) the *interpatch* shearing time $\bar{\tau}_n^s=\langle(\partial v/\partial x)^2\rangle^{-1/2} \sim \beta_n^{-1/2}\tau_n^s > \tau_n^s$ ($\langle\,\rangle$ denotes a spatial average), (iii) the collapse time τ_c, and (iv) the dissipation time $\tau_n^d=l_n^2/\nu$, where ν is the viscosity. The interpatch shearing time increases with decreasing scale size, and is thus dynamically irrelevant since it is less efficient in decomposing clouds than local tidal disruptions. To account for the fact that the clouds are compressed at each stage, the Kolmogorov argument needs to be modified to consider energy transfer per unit volume, i.e.,

$$\epsilon=\beta_n\, dE_n/dt \sim \beta_n\rho_n v_n^2/\tau_n^s=\bar{\epsilon}=\text{const},$$

with the quasivirialization relation, $\rho_n \sim v_n^2/4\pi G l_n^2$, entering as a constraint. We then obtain

$$v_n/v_0 \sim (l_n/l_0)^{(3-\mu)/5}, \quad \rho_n/\rho_0 \sim (l_n/l_0)^{-2(2+\mu)/5}, \qquad (2)$$

where $v_0=(4\pi G\bar{\epsilon}l_0^3)^{1/5}$ and $\rho_0=[\bar{\epsilon}^2 l_0/(4\pi G)^3]^{1/5}$ are, respectively, the velocity shear and density fluctuation on the largest scales of the hierarchy. The stronger spatial correlations manifested here relative to the incompressible case are due to the added constraint imposed by quasivirialization. That the latter can be interpreted as an internal energy injection mechanism is sustained upon noting that the spectral quantity[14] $\tilde{E}(k)\equiv \rho v^2/K \propto k^{-7/5}$ is flatter (i.e., more energy content at each scale size) relative to the corresponding incompressible (ρ=const) energy spectreum $E(k)\propto k^{-5/3}$. The v^2 spectrum itself, which is proportional to $k^{-(11-2\mu)/5}$, shows remarkable resemblance to a spectrum of random shocks ($\propto k^{-2}$). It remains to determine the fractal dimension D. In the absence of a systematic procedure, a bound on D may be obtained with the information-theoretic arguments of Mori and Fujisaka.[15] Identifying the fraction of space occupied by *all* the n-clouds with the probability $\mathcal{I}$ that a given volume is active, it is postulated that this probability takes on a value such as to maximize the information entropy of intermittency, defined by $\mathcal{H}=-\mathcal{I}\ln\mathcal{I}-(1-\mathcal{I})\ln(1-\mathcal{I})$. The result is $\mu=\frac{5}{6}\log_{\mathcal{N}}(2\mathcal{N}^{2/5}-1)-\frac{1}{3}$, where $\mathcal{N}^D=(l_{n-1}/l_n)^D$ is the number of offspring clouds per generation. Since $\infty \ge \mathcal{N} \ge 1$, it follows that $0\le\mu\le\frac{1}{3}$ or $\frac{8}{3}\le D\le 3$. This is, at best, consistent with the observational evidence that clouds have a filamentary or ribbonlike structure (i.e., $3>D>2$), but any more quantitative comparison would be unwarranted. Finally, the theory allows us to estimate the spectral mass function $N(m)$, defined as the number of clouds with mass between m and $m+dm$:

$$\int_{m_{n-1}}^{m_n} N(m)dm = N_n,$$

where N_n is the number of clouds between spatial size l_n and l_{n-1}. We find $N(m) \propto m^{-\alpha}$, where $\alpha = 1 + (13 - 6\mu)/(11 - 7\mu) \simeq 2.2$. This value is somewhat steeper than inferences made from observational data ($1 < \alpha < 2$) (Ref. 6); such inferences, on the other hand, are open to interpretation, and have large error bars associated with them.

The physics of collisionless plasmas is described by the Vlasov equation augmented by Poisson's equation to ensure self-consistency:

$$\begin{aligned} df/dt &= \partial f/\partial t + v\,\partial f/\partial x - (e/m)E\,\partial f/\partial v = \mathcal{C}(f), \\ \partial E/\partial x &= 4\pi n e\left(1 - \int dv f\right), \end{aligned} \tag{3}$$

where f is the normalized electron phase-space distribution function, E is the electrostatic field, $\mathcal{C}(f)$ is the Fokker-Planck collision operator [$\mathcal{C}(f)=0$ for a collisionless plasma; we include it here to motivate the termination of the cascade process], n is the density, and e and m are the electron charge and mass, respectively. Immobile ions assume only a background-neutralizing role and affect dynamics only through Poisson's equation. With the assumption of a phase-space density fluctuation of size $\tilde{f}$, with velocity spread Δv and center-of-mass speed $u < v_t$ (v_t is the thermal speed), a dispersion relation can be straightforwardly derived from Eq. (3):

$$(\omega - ku)^2 = k^2(\Delta v)^2 + \omega_p^2 \tilde{f} \Delta v/\epsilon(k,ku), \tag{4}$$

where $\epsilon(k,\omega) = 1 + (\omega_p^2/k)\mathcal{P}\int dv\,\partial f_0/\partial v(\omega - kv)^{-1}$ is the plasma dielectric function which shields fluctuations, f_0 is the equilibrium distribution function, and $\mathcal{P}$ denotes a principal value. Transforming to the fluctuation rest frame, the Jeans-instability criterion for a collisionless plasma becomes

$$\gamma^2 = -\omega_p^2 \tilde{f} \Delta v/\epsilon_0 - k^2(\Delta v)^2, \tag{5}$$

where $\epsilon_0 = \epsilon(k,0) = 1 + (k\lambda_D)^{-2}$, and $\lambda_D \sim v_t/\omega_p$ is the Debye shielding length. Equation (5), which should be compared with Eq. (1), expresses the competition between the self-trapping tendency of negative fluctuations or phase-space density holes ($\tilde{f} < 0$), and the shearing action of repulsive electrostatic forces. The balance between these two forces, i.e., $\tilde{f} = -\epsilon_0 \Delta v/\omega_p^2(\Delta x)^2$, is the phase-space analog of quasivirialization. For a turbulent distribution of phase-space density structures to be able to coexist, it is necessary that a given hole not be tidally disrupted under the electrostatic straining deformations of all the other holes. Implicitly assumed in the theory, therefore, is that the self-trapping time $\tau_{tr} = (\Delta x/\Delta v)_{self}$ is comparable to the turbulent decorrelation (clump[2]) time $\tau_{cl} = (\Delta x/\Delta v)_{turbulent}$. As we are interested in fluctuations with initial spatial dimension larger than the Debye length (it is only for such fluctuations that self-trapping is significant[2]), the shielding term effectively truncates all scales $\Delta x > \lambda_D$ to λ_D and the quasivirialization condition reduces to the simple form $\tilde{f} = -\Delta v/v_t^2$.

Equation (3) expresses the conservation of phase-space density fluctuations along particle trajectories. The mean-square fluctuation density is therefore a quadratic invariant of the collisionless system. When left to evolve self-consistently under their mutual interactions, phase-space density fluctuations "fissure" by ballistic streaming and velocity-space diffusion. This is best illustrated by the evolution equation for the two-point correlation function[1,2]:

$$(\partial_t + v_- \partial_{x-} - \partial_{v-}\mathcal{D}_- \partial_{v-} - \mathcal{C})\langle \delta f(1)\delta f(2)\rangle = 0,$$

where

$$\mathcal{D}(x_-) = (e/m)^2 \int dk[1 - \cos(kx_-)] g_{k,kv} \langle |E|^2\rangle_{k,kv} \sim \langle \Delta v^2\rangle/\tau$$

is the velocity-space diffusion coefficient, $g_{k,\omega}$ is the renormalized temporal propagator, and $(x_-,v_-) = (x_1 - x_2, v_1 - v_2)$ is the phase-space separation between the two points (x_1,v_1) and (x_2,v_2). In words, successively smaller scale fluctuations will be generated through electrostatic straining deformations and consequent phase-space gradient amplification of hole structures. This is an autonomous process in the sense that it evolves self-consistently without the need for an external free-energy source. There is an alternative perspective on the problem, originated by Saffman in the context of 2D Navier-Stokes turbulence,[16] which focuses on discontinuities in the fluctuations as providing the dominant contribution to the inertial range spectrum. Indeed, phase-space advection can be expected to bring chunks of fluid of varying f together at neighboring points, leading to the creation of quasidiscontinuous boundary layers. This argument leads to a k^{-2} spectrum for phase-space density fluctuations, and can be expected to determine the very early phase of the temporal evolution. However, the incompressibility of the flow precludes the interpenetration of phase-space quasidiscontinuities, and as they pile up, the longer-time evolution can be expected to proceed by a cascading of phase-space density fluctuations from large to small scales.

Assuming a hierarchy of quasivirialized (i.e., self-trapped) phase-space density holes, we assume a stationary process whereby the fluctuation density available in large holes is transferred self-similarly to small holes at a rate characterized by the inverse trapping time, $\tau_n = \Delta x_n/\Delta v_n$. Since the nonlinear cascade can neither create nor destroy fluctuation density, the rate of transfer must be constant and given by $\zeta = \tilde{f}_n^2/\tau_n = \text{const}$. As in the case of compressible self-gravitating matter, the cascade is constrained by quasivirialization. This constraint, which implicitly "slaves" the spatial and velocity dimensions of phase space to each other, suggests that the cascade can equivalently be thought of as a successive sequence of reduced spatial scales, or as one of reduced velocity scales. The cascade terminates when the nonlinear transfer term becomes comparable to the Fokker-Planck term, which in magnitude is of order $\nu(v_t/\Delta v)^3 f$, where ν is the electron collision frequency. A Reynolds-type number, Ξ, characterizing the ratio of dissipative to nonlinear trapping time scales, can then be usefully defined: $\Xi = (\Delta v)^4/\nu v_t^3 \Delta x$. In terms of the free parameters of the system, the dissipative scale may thus be determined: $(\Delta x_d, \Delta v_d) = (\nu^3/\zeta^4 v_t^7, \nu/\zeta v_t)$, and the number of excited degrees of freedom (ignoring the possibility of attractors) is given by $\Delta x_0 \Delta v_0/\Delta x_d \Delta v_d = \Xi^4$. In the inertial range, where dissipation is subdominant to the nonlinear coupling term (i.e., $\Xi \gg 1$), we obtain the following scaling laws:

$$\Delta v_n/\Delta v_0 \sim (\Delta x_n/\Delta x_0)^{1/3}, \quad \tau_n/\tau_0 \sim (\Delta x_n/\Delta x_0)^{2/3},$$
$$F(k) \sim (\zeta/v_t^2)^{2/3} k^{-5/3}, \tag{6}$$

where $\Delta v_0 \sim (\zeta v_t^4 \Delta x_0)^{1/3}$, $\tau_0 \sim [(\Delta x_0)^2/\zeta v_t^4]^{1/3}$, and $F(k)$ is the spectral fluctuation density. The resemblance to the Kolmogorov scaling in 3D Navier-Stokes turbulence is purely coincidental and a direct outcome of the quasivirialization constraint. Other hole properties, such as hole mass and energy spectrum can be easily derived: $m_h \sim (\zeta v_t^2)^{2/3} \Delta x^{5/3}$ and $E_h(k) \sim mn(\zeta^2 v_t^5)^{2/3} k^{-10/3}$. It should be borne in mind that neither hole mass nor hole energy is conserved from one hierarchy level to the next, but in fact is lost to the interstitial region between holes in the process of hole splitting.

A novel feature of this hole mitosis model is its intrinsic intermittency and anisotropy. This is due to the fact, mentioned earlier, that the velocity dimension of the hole structure is slaved to its spatial dimension through the quasivirialization constraint. Thus, for example, if a hole halves its spatial extent, its width in velocity space must adjust by a factor of $2^{1/3} < 2$ in order for it to remain self-trapped. An immediate consequence is that succeeding generations of holes fill less and less of the area in phase space available to them, and an initially space-filling distribution of large holes evolves into a patchy, anisotropic pattern of quasi-one-dimensional structures whose velocity dimension is highly elongated relative to their spatial extent. This explains the numerical observation that skewness, defined by $S = \langle f^3 \rangle / \langle f^2 \rangle^{3/2}$, increases both in time and with decreasing scale, signifying increasing deviation from a Gaussian distribution. Semiquantitatively, if we choose to adopt $\frac{4}{3}$ as a convenient definition of the dimension of phase-space holes (hole area $= \Delta x \Delta v \propto \Delta x^{4/3}$), and follow through with arguments similar to those given in the first part of this Letter, we obtain a self-similarity dimension D equal to unity, as suggested by the preceding physical argument. The intrinsically non-space-filling character of hole evolution is indeed what is responsible for the noninvariance of hole mass and energy.

In summary, we have presented in this Letter a simple dynamical model of hole mitosis to account for the early evolution of phase-space density fluctuations in a Vlasov plasma. The long-time evolution, which is characterized by coalescence phenomena, is not addressed by this picture. The model, the essential ingredient of which is a cascade constrained by the physical requirement of quasivirialization or self-trapping, exhibits interesting geometric properties such as intrinsic intermittency and anisotropy. A variant of the model holds promise for explaining certain features of compressible self-gravitating matter in the ISM. Further details of the latter will be presented in a forthcoming publication. It should be remarked that while these ideas do not necessarily carry over to 3D unmagnetized plasma turbulence, they can well be applicable to a strongly magnetized situation *in the direction along the field*; $\mathbf{E} \times \mathbf{B}$ transport would then be expected to determine cross-field plasma dynamics.

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