# Geometric Canonical Phase Factors and Path Integrals

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(Received 29 February 1988)

It is shown that the geometric phase accompanying an arbitrary cyclic change of the state vector can be naturally understood as a canonical phase term in the coherent-state path integral. The adiabatic phase is shown to be derived as a part of the canonical phase.

PACS numbers: 03.65.Bz, 03.65.Sq

The geometrical or topological phase factor accompanying an adiabatic cyclic change has had an impact on diverse areas in quantum physics.[1] It was originally introduced in connection with the intersection of molecular energy surfaces[2,3] and recently formulated in a general quantum mechanical framework with use of the standard formulation[4,5] or the path-integral method.[6] This phase factor has been experimentally detected by the use of, for example, molecular spectra[7] and optical phenomena.[8]

Very recently Aharonov and Anandan have extended the concept of this phase to the case where the cyclic change takes place in a general manner without recourse to adiabaticity.[9] Garrison and Chiao have applied this viewpoint to classical nonlinear wave phenomena.[10]

In this Letter we shall proceed a step further towards obtaining the geometric phase associated with general cyclic evolution. This is achieved by use of the path integral in the coherent-state representation.[11-13] As a consequence of this investigation we get a new look at the generalized geometric phase, which yields the adiabatic phase in a novel fashion.

We start with a concise description of the phase change associated with cyclic quantum evolution.[9,10] This is simply defined as the generalized Hill-Bloch law:

$$|\psi(t+T)\rangle = \exp(i\Phi)\,|\psi(t)\rangle. \tag{1}$$

Here $|\psi(t)\rangle$ is the normalized state vector satisfying the time-dependent Schrödinger equation

$$[i\hbar\,\partial/\partial t - \hat{H}(t)]\,|\psi(t)\rangle = 0.$$

The appearance of the phase $\Phi$ can be assured by the unitarity of the evolution operator $U(T,0)$, the eigenvalue of which is simply given as $\exp(i\Phi)$. In order to evaluate $\Phi$, we write $|\psi(t)\rangle = \exp[-if(t)]\,|\tilde{\psi}(t)\rangle$, where $|\tilde{\psi}(t)\rangle$ satisfies the periodicity condition $|\tilde{\psi}(t+T)\rangle = |\tilde{\psi}(t)\rangle$. Using the Schrödinger equation we get

$$df/dt = \langle\tilde{\psi}\,|\,i\hbar\,\partial/\partial t - \hat{H}(t)\,|\,\tilde{\psi}\rangle,$$

and thus

$$\Phi = \int_0^T \langle\tilde{\psi}\,|\,i\hbar\,\partial/\partial t\,|\,\tilde{\psi}\rangle dt - \int_0^T \langle\tilde{\psi}\,|\,\hat{H}\,|\,\tilde{\psi}\rangle dt$$
$$= \gamma + \delta. \tag{2}$$

The first term of (2) represents the geometric phase.

The formula (2) is simple enough that one can appreciate its meaning. If we take the adiabatic state for $|\tilde{\psi}(t)\rangle$, we get the adiabatic phase. However, there may be a critical question in the above derivation, the question of how one should choose the explicit form of $|\tilde{\psi}(t)\rangle$. The crucial point of the above derivation is the use of the indefiniteness of the phase inherent in the state vector. Alternatively, it may be possible to deduce the geometric phase $\gamma$ in a constructive way without invoking the phase ambiguity. In what follows we shall give an answer to this problem.

First we shall rewrite Eq. (1) as

$$\langle\psi_0\,|\,U(T,0)\,|\,\psi_0\rangle = \exp(i\Phi). \tag{3}$$

This means that the phase under consideration is nothing but the transition amplitude for the quantum transition from the initial state $|\psi_0\rangle$ to the same final state. The evolution operator is written as the time-ordered product,

$$U(T,0) = T\exp\left[-i\int_0^T \hat{H}(t)dt/\hbar\right]$$
$$= \prod_{k=1}^{\infty} \exp[-i\hat{H}(k)\epsilon/\hbar], \tag{4}$$

where $\hat{H}(k)$ denotes the Hamiltonian at $t=t_k$. This involves the time-independent system as a special case, for which the evolution operator becomes $U(T,0) = \exp(-i\hat{H}T/\hbar)$. We shall consider the quantum evolution of the system in a Hilbert space labeled by an appropriate complex parameter symbolically written as $\{|Z\rangle\}$. We impose a specific property on this space, namely, the "partition of unity" (see below). The Hilbert space possessing this property may be naturally realized as the "generalized coherent state" [simply called coherent state (CS)]. The CS is defined on the basis of the unitary representation of a Lie group[14] for which we shall give a quick sketch. Let $T(g)$ be the irreducible unitary representation of the Lie group $G$ and consider the set $\{|g\rangle = T(g)\,|0\rangle\}$, where $|0\rangle$ is some starting vector. We define the subset $H$ of $G$ such that for $h \in H$ the relation $T(h)\,|0\rangle = (\text{phase})\,|0\rangle$ holds. $H$ forms a subgroup of $G$ which is called the isotropy group. Now, for a given point $Z$ of the quotient space $G/H$ (which is assumed to be a complex homogeneous space), we can arbitrarily choose a $g \in G$ belonging to the coset corresponding to $Z$. This choice is not unique, since $g' = gh$, for any $h \in H$, is in the same coset. This corresponds to the choice of a section of the fiber bundle $(H, G, G/H)$. Then we write $|Z\rangle = |g\rangle$; namely, for each $Z$ we can assign a particular state just called the coherent state. From this construction we see that the set $\{|Z\rangle\}$ forms a section of the line bundle over $G/H$. It is to be noted that the line bundle is *nontrivial* and the section is not continuous in general. Furthermore, the set of CS forms a so-called "overcomplete set" satisfying the nonorthogonal normalization condition, namely, $|\langle Z'|Z\rangle| \leqq 1$, where the equality holds for the case that $Z' = Z$. The characteristic property of the CS thus constructed is the partition of unity:

$$\int |Z\rangle d\mu(Z) \langle Z| = 1, \tag{5}$$

with the invariant measure $d\mu(Z)$ on $G/H$. Here the existence of the measure is not necessary. If $G$ is compact it exists; otherwise, the integral may diverge. It should be noted that a *sequence* of coherent states, say $\{|Z^{(1)}\rangle, \dots, |Z^{(k)}\rangle, \dots, \}$, is obtained according to the choice of the initial state $|0\rangle$. Usually the sequence just corresponds to the inequivalent class of irreducible representations; e.g., for the case of $G = \mathrm{SU}(2)$ it is labeled by the magnitude of spin. Two coherent states may be "unconnected" if they belong to different "classes" of this sequence.

Now we define the transition amplitude for the cyclic change in the CS space:

$$K(T) = \langle Z_0| \prod_{k=1}^{\infty} \exp[-i\hat{H}(k)\epsilon/\hbar] |Z_0\rangle, \tag{6}$$

where $\hat{H}(t)$ is given by a function of the generators of $G$ only, e.g., a linear or quadratic function, and $|Z_0\rangle$ is prescribed to belong to a specific class of the CS sequence. We insert the completeness relation (5) between each product of (6). Thus we have the transition amplitude of the type

$$\langle Z_{k+1}^{\beta} | \exp[-i\hat{H}(k)\epsilon/\hbar] | Z_k^{\alpha} \rangle$$

for the infinitesimal time interval; namely, it represents the transition from a CS belonging to one class $(\alpha)$ to a CS belonging to another class $(\beta)$. The transition occurs only between CS's belonging to the same class, i.e., $\alpha = \beta$, since the matrix element between the states belonging to different classes vanishes for the case of the Hamiltonian considered here. In the following we omit the index $\alpha$. In the limit of $\epsilon \to 0$, we have an expansion

$$\begin{aligned} \langle Z_{k+1} | \exp[-i\hat{H}(k)\epsilon/\hbar] | Z_k \rangle &\cong \langle Z_{k+1} | Z_k \rangle - i(\epsilon/\hbar) \langle Z_{k+1} | \hat{H}(k) | Z_k \rangle \\ &\approx \langle Z_{k+1} | Z_k \rangle \exp[-i(\epsilon/\hbar) \langle Z_{k+1} | \hat{H}(k) | Z_k \rangle / \langle Z_{k+1} | Z_k \rangle], \end{aligned} \tag{7}$$

and hence

$$K(T) = \int \prod_{k=1}^{\infty} d\mu(Z_k) \prod_{k=1}^{\infty} \langle Z_{k+1} | Z_k \rangle \exp[-i(\epsilon/\hbar) \langle Z_{k+1} | \hat{H}(k) | Z_k \rangle / \langle Z_{k+1} | Z_k \rangle]. \tag{8}$$

Geometrically, the overlap function $\langle Z_{k+1} | Z_k \rangle$ represents the parallel transport of the CS vector at the point $Z_k$ to the CS vector at the nearby point $Z_{k+1}$. In the gauge-field language, it defines the "connection field" between two nearby points $Z_{k+1}$ and $Z_k$. Thus, the infinite product in (8) gives the finite connection along a loop $C$ given by division points $\{Z_k\}$.[15] This is just regarded as an extension of the adiabatic connection which is defined as the overlap between neighboring adiabatic levels with the quantum number $n$, i.e., $\langle n(X_{k+1}) | n(X_k) \rangle$.[6] Now, using the expansion[16]

$$\langle Z_{k+1} | Z_k \rangle \cong 1 - i \langle Z_k | i\hbar\, \partial/\partial t | Z_k \rangle (\epsilon/\hbar) \cong \exp[i(\epsilon/\hbar) \langle Z_k | i\hbar\, \partial/\partial t | Z_k \rangle], \tag{9}$$

we get the path-integral expression for (8):

$$\langle Z_0 | T \exp[-i \int_0^T \hat{H}(t) dt/\hbar] | Z_0 \rangle = \int \exp[iS(C)/\hbar] \prod_t d\mu(Z(t)), \tag{10}$$

with the phase term

$$\begin{aligned} S &= \int_0^T \langle Z | i\hbar\, \partial/\partial t - \hat{H}(t) | Z \rangle dt \\ &= \Gamma + \Delta. \end{aligned} \tag{11}$$

Equation (10) is an extended form of Eq. (2), namely, the summation over all closed loops $C$. The first term $\Gamma$ gives the geometric phase. Equation (10) is nothing but the coherent-state path integral previously studied in several forms, e.g., the spin coherent state[11] and unitary coherent state.[12,13] From the path-integral aspect, the phase $S$ is nothing but the action function; in particular, the first term of (11) is the one that should be called the *canonical term*, and so the geometric phase is alternatively called the canonical phase. In this way we have nothing new from the path-integral viewpoint, but we have arrived at a new insight into the geometric phase for general cyclic change. We shall take this a step further.

We shall examine the connection with the conventional expression (2). We can get the principle for choosing the loop $C$ from the path-integral expression (10), whereas in (2) we have no way to fix the form of $|\tilde{\psi}(t)\rangle$. This may be simply obtained as a consequence of the semiclassical limit $\hbar \rightarrow 0$; namely, the method of stationary phase yields the simple exponential form

$$\langle Z_0 | U(T,0) | Z_0 \rangle = \exp[i\Gamma(C)] \exp\left[ -\frac{i}{\hbar} \int_0^T \langle Z | \hat{H}(t) | Z \rangle dt \right]. \tag{12}$$

The exponent of (12) has the same form as Eq. (2). However, the closed loop $C$ is determined by the periodicity condition $Z(t+T)=Z(t)$ together with the variation equation $\delta S=0$ which yields the equation of motion in the complex parameter space.[13] This is just what determines the geometric phase. We note that (12) may be regarded as a counterpart of the adiabatic phase change and this fact implies that the topological and geometrical structure emerges as a semiclassical limit of the coherent-state path integral.

Now we consider typical applications of the general theory. The first example is the spin system which is described by the SU(2) CS, the base space of which becomes SU(2)/U(1) = Bloch sphere. The SU(2) CS is given by

$$|Z\rangle = (1+|Z|^2)^{-J} \exp(Z\hat{J}_+)|0\rangle, \tag{13}$$

where $(\hat{J}_\pm, \hat{J}_z)$ denotes the spin algebra and $|0\rangle = |J,-J\rangle$ denotes the lowest member of the irreducible representation with weight $J$. The canonical phase thus becomes

$$\oint \langle Z | i\hbar\, \partial/\partial t | Z \rangle dt = \int i\hbar J (1+|Z|^2)^{-1} (Z^* \dot{Z} - \text{c.c.}) dt. \tag{14}$$

The integral is taken along the path determined by the equation of motion

$$2iJ\hbar\, \partial Z/\partial t = (1+|Z|^2)^2 \partial H/\partial Z^*, \quad H = \langle Z | \hat{H} | Z \rangle,$$

together with the periodic boundary condition $Z(t+T) = Z(t)$. If we use the Stokes theorem, Eq. (14) is converted to the surface integral which is just equal to the area surrounded by a closed orbit on the Bloch sphere. I expect that this area may be closely related to the nontriviality of the line bundle. The extension to a more general unitary CS may also be interesting; the base space is the complex projective space $P_n(C)$ and the canonical phase is expressed as an integral of the differential two form induced by the so-called Fubini-Study metric. These will be discussed elsewhere.

In the second example we examine how the adiabatic phase can be derived as a special case of the general geometric phase. In order to achieve this we note that the coherent state $|Z\rangle$ gives an approximate state vector for the many-particle Hamiltonian.[12,13] Let us suppose that the degrees of freedom may eventually be separated into two parts, say, "external" and "internal" degrees of freedom which may be described by $\xi$ and $q$, respectively, and the Hamiltonian is given by $\hat{H} = \hat{H}_0(\xi) + \hat{h}(q,\xi)$. If the external degrees of freedom can be treated as slow variables, the approximate form of the state vector would be written as a tensor product

$$|Z\rangle = |\xi\rangle \otimes |n(\xi)\rangle, \tag{15}$$

where $|n(\xi)\rangle$ stands for the normalized adiabatic level of the internal Hamiltonian $\hat{h}(q,\xi)$ when the external variable takes the value $\xi$ and the adiabatic energy is $\lambda_n$. Thus the canonical phase becomes

$$\int \langle Z | i\hbar\, \partial/\partial t | Z \rangle dt = \int \langle \xi | i\hbar\, \partial/\partial t | \xi \rangle dt + \int \langle n(\xi) | i\hbar\, \partial/\partial t | n(\xi) \rangle dt. \tag{16}$$

Hence, we get

$$S = \int_0^T [\langle \xi | i\hbar\, \partial/\partial t | \xi \rangle - H_0 - \lambda_n + \langle n(\xi) | i\hbar\, \partial/\partial t | n(\xi) \rangle dt. \tag{17}$$

The first three terms of (17) give the external action function and the last term represents the adiabatic phase which becomes the topological action to be added to the external action function.[6] If $|\xi\rangle$ is eventually given by the coherent state, e.g., boson CS for which $\xi$ becomes the canonical pair $(Q,P)$, the first term of (16) yields the kinematical Pfaff form, whereas the second term leads to the topological phase. In this way the canonical phase would involve both kinematical and topological structure simultaneously, which is a significant aspect of the canonical phase.

The author would like to thank Dr. Peter Davis for his careful checking of the manuscript and Dr. Shinji Iida for his useful comments. This work was carried out under the auspices of the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, and Culture No. 62540216.

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[16]Here we assume that the connection function $\langle Z_{k+1} | Z_k \rangle$ is close to 1 for almost all paths contributing to the path integral. Actually, this assumption breaks down where paths cross any discontinuous points of the section of line bundle. However, this catastrophic feature may be canceled out in the final path-integral expression.