Cascade Focusing in the Beat-Wave Accelerator

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The 2D wave-envelope equations for the beat-wave–cascade system are studied analytically and numerically. An expression for the mean square width of the cascade envelope is obtained, and is used to predict the long-term behavior of the waves. The amplitude of a resonantly driven plasma wave falls significantly over a stage length because of enhanced diffraction of the cascade envelope. Conversely, detuning the pumps from the plasma frequency can lead to *focusing* of the envelope and a corresponding increase in plasmon amplitude of up to 200% over the same distance.

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Much of the recent interest in the beat-wave accelerator concept[1] has concentrated on the evolution and saturation of the plasma wave generated by the beating of two copropagating laser beams.[2-6] Experiments on beat-wave excitation by Clayton *et al.*[7] and Ebrahim, Lavigne, and Aithal[8] have produced evidence for longitudinal electric fields of 1 GV m^{-1}, though the difficulty of diagnosing the plasmon amplitude has prevented quantitative comparison with fluid theory. Plasmon generation has also been demonstrated with particle simulations,[9] which have indicated the importance of 2D effects on both plasmon dynamics and laser-pulse propagation.

The interaction between the plasmon and the electromagnetic (em) waves has also been studied by several authors.[6,10] Karttunen and Salomaa[6] derived 1D analytical solutions for both the plasmon and the Raman cascade sidebands and showed how the fluctuations in the em spectrum could be used as a diagnostic for the plasma wave. In this Letter, we formulate the plasmon-cascade problem in 2D. We derive an analytical expression for the mean square width of the cascade envelope. The expression is evaluated for various laser parameters and is used to predict the long-term transverse behavior of the em waves, and consequently the effects of cascading on plasmon amplitude and phase. These results are compared to numerical solutions of the 2D envelope equations, and we thus show how a suitable choice of laser pump parameters can improve the "quality" of the plasmon over a stage length.

We consider two collinear electromagnetic waves described by vector potentials $A(\omega_0,\mathbf{k}_0)$ and $A(\omega_1,\mathbf{k}_1)$ propagating in the z direction through an infinite plasma slab. If the matching conditions $\omega\equiv\omega_1-\omega_0=\omega_p$ and $\mathbf{k}_1-\mathbf{k}_0=\mathbf{k}_p$ are satisfied, then a beat plasmon with phase velocity $v_p=c(1-\omega_p^2/2\omega_0\omega_1)$ is generated. We assume that the plasma is well underdense ($\omega_0/\omega_p=100$–300), so that $v_p\simeq c$, and consequently the plasmon will not suffer significant Landau damping.

Starting from Maxwell's equations in planar (z,x) geometry, and using the wave envelopes

$$A(z,x,t)=\tfrac{1}{2}\textstyle\sum_m[a_m(z,x,t)\exp\{i(k_mz-\omega_mt)\}+\text{c.c.}]$$

for the electromagnetic vector potential, and

$$E(z,x,t)=\tfrac{1}{2}[e(z,x,t)\exp\{i(k_pz-\omega t)\}+\text{c.c.}]$$

for the plasma electric field, we obtain the following equations for the electromagnetic cascade and plasmon, respectively:

$$\omega_m\frac{\partial a_m}{\partial t}-\frac{i}{2}\frac{\partial^2 a_m}{\partial x^2}=\frac{k_p}{4}(e^*a_{m+1}-ea_{m-1}),\qquad(1)$$

$$-v_p\frac{\partial e}{\partial z}+i\delta e-\frac{3i}{16}|e|^2e=\frac{fk_p}{4}\sum_m a_ma_{m-1}^*.\qquad(2)$$

The time and space variables in Eqs. (1) and (2) are normalized to ω^{-1} and c/ω, respectively. Likewise, all frequencies and wave numbers are normalized to ω and ω/c; the detuning parameter in the plasma equation is given by $\delta=(f-1)/2$, and the linear dispersion relation is $\omega_m^2=k_m^2+f$, where $f\equiv\omega_p^2/\omega^2$ and $\omega_m=\omega_0+m$. The plasma electric field in these units is given by $e=qE/m\omega c$; the electromagnetic vector potential by $a=qA/mc$.

Equations (1) and (2) have been simplified by our making a coordinate transformation to a frame moving at the phase velocity v_p of the plasmon: $t=t'$, $z=z'+v_pt'$. (The primes have been omitted.) We have neglected $\partial_{t'}e$ compared to $v_p\,\partial_{z'}e$ since the time scale is now determined by the evolution of the cascade envelope equation, namely, $\tau_{\text{em}}=\omega_0^{-1}L_x^{-2}$. We have also omitted terms in both equations arising from the relativistic quiver motions in the em field, and a term $|e|^2a_m/8$ in Eq. (1). This approximation is valid for irradiances $I_0\lambda^2<10^{16}$ W cm^{-2} μm^2, where we are below the respective focusing thresholds due to these two effects. The collision frequency is reduced by the electron quiver motion in the plasma wave (an effect considered, for example, in Ref. 11) and we may therefore neglect collisional damping provided that $|e|_{\max}\gg v_{\text{th}}/c$.

In arriving at Eq. (2) we have neglected thermal effects[6] (which introduce a term like $v_{\text{th}}^2\,\partial_z e$), and 2D plasma-wave nonlinearities. The latter omission is valid provided that the envelope width is much larger than a collisionless skin depth, in which case the radial field $e_x\ll e_z$ and we may also neglect the self-consistent B

field.[12] We have also neglected wave-number mismatches, which introduce terms $O(\omega_0^{-3})$, and are therefore small over an accelerator stage length[1] $L_{dp} \simeq 3\omega_0^2$. The effects of ion motion[13] have been omitted. The higher-order couplings to plasma-wave harmonics at $2\omega_p$, $3\omega_p$, etc., have growth rates $\tau^{-1} < |e|^2 \tau_c^{-1}$, where τ_c^{-1} is the cascade growth rate. We may therefore neglect these couplings for small plasmon amplitudes ($|e| \sim 10\%$).

By taking moments of the em equations (1), we may obtain useful information concerning the behavior of the cascade envelope. In particular, we consider the mean square "separation" of the latter, defined by[14]

$$\langle \delta x^2 \rangle = I^{-1} \sum_m \int (x^2 - \langle x \rangle^2) \omega_m |a_m|^2 d^3x.$$

The symmetry of the pumps implies $\partial_t \langle x \rangle = 0$. Using this result and the conservation of total wave action $I \equiv \int \sum \omega_m |a_m|^2 d^3x$, we may write

$$\frac{\partial^2}{\partial t^2} \langle x^2 \rangle = I^{-1} \frac{\partial^2}{\partial t^2} \sum_m \int x^2 \omega_m |a_m|^2 d^3x,$$

which after some algebra reduces to

$$\frac{\partial^2}{\partial t^2} \langle x^2 \rangle = I^{-1}(D+C), \qquad (3)$$

where

$$D = 2 \int \sum_m \omega_m^{-1} \partial_x a_m \, \partial_x a_m^* \, d^3x, \qquad (4)$$

$$C = -\mathrm{Im} \left[\int \sum_m \omega_m^{-1} a_m^* a_{m+1} x \frac{\partial e}{\partial x} d^3x \right]. \qquad (5)$$

Similarly, we find that the mean square length of the cascade envelope $\langle z^2 \rangle = \text{const}$.

In the absence of coupling to the plasma wave ($C=0$), D is constant in time, and we retrieve the usual result for a laser envelope $\langle x^2 \rangle = Dt^2/2I + Ft + \langle x_0^2 \rangle$, where F and x_0 are constants of integration.[15] The second term is zero for initially parallel wave fronts.

When cascading is present, D and C are both time dependent, and so we cannot integrate Eq. (3) for nonzero C without first solving the envelope equations. However, we can evaluate $\partial_{tt} \langle x^2 \rangle$ at $t=0$, given the boundary conditions $a_m = 0$, $m \neq 0,1$, and $a_{0,1} = \alpha_{0,1} \times \exp(-x^2/\sigma^2 - z^2/\tau^2)$, where $\alpha_{0,1}$ are the normalized peak quiver velocities in the laser fields. In the regime $c/\omega_p \ll \tau \ll \tau_{RL}$, where τ_{RL} is the Rosenbluth-Liu saturation period,[2] the plasmon envelope field is almost wholly real, and we have $C=0$. For resonantly driven plasmons with $\tau \sim \tau_{RL}$, the phase varies between 0 and π during the buildup and saturation period, resulting in $C > 0$. In order to compare the diffraction and cascading effects, we evaluate D and C for various pump intensities I_0 and widths σ. The plasmon amplitude e is found from Eq. (2) with $\tau = \tau_{RL}$ and $\delta = 0$. We may then express the width σ_b for which cascading is the dominant mechanism for beam spreading ($C \geq D$) as

$$\frac{\sigma_b}{c/\omega_p} \geq 18.3 \left[\frac{I_0 \lambda_0^2}{10^{15}\ \mathrm{W\,cm^{-2}}\ \mu\mathrm{m}^2} \right]^{-1/6}. \qquad (6)$$

We now consider the effect of detuning the beat frequency ω from ω_p such that the frequency mismatching opposes the relativistic frequency shift. This yields a higher saturation value for e_z (Refs. 3 and 6) but the phase now varies between $-\pi$ and π, and is negative over a large part of the plasmon buildup region. Numerical evaluation of $\mathrm{Im}(e_z)$ as a function of the density mismatch $\Delta n/n = 2\delta$ yields the optimum detuning for minimum C as

$$(\Delta n/n)(\mathrm{opt}) = 0.007 \left[\frac{I_0 \lambda_0^2}{10^{15}\ \mathrm{W\,cm^{-2}}\ \mu\mathrm{m}^2} \right]. \qquad (7)$$

As before for the resonant case, we can evaluate $A = D + C$ at $t=0$, keeping $\tau = \tau_{RL}$ and taking $(\Delta n/n)(\mathrm{opt})$ from Eq. (7). We therefore obtain the condition for cascade focusing ($C + D < 0$):

$$\frac{\sigma_b}{c/\omega_p} \geq 14.6 \left[\frac{I_0 \lambda_0^2}{10^{15}\ \mathrm{W\,cm^{-2}}\ \mu\mathrm{m}^2} \right]^{-1/6}. \qquad (8)$$

To investigate the long-term behavior of the plasmon-cascade system, we integrate Eqs. (1) and (2) numerically. The scheme is fully implicit for both the plasmon and cascade sidebands, and is formulated to conserve wave action I. This ensures that energy conservation also holds to a good approximation—particularly for large frequency ratios $\omega_0/\omega_p > 100$. We consider the case of (i) a resonantly driven plasmon ($D + C > 0$) and (ii) an optimally detuned plasmon ($D + C < 0$).

In both cases the laser wavelengths are $\lambda_1 = 1.053\ \mu$m, $\lambda_0 = 1.064\ \mu$m, and the plasma frequency is $\omega_p = 1.85 \times 10^{13}\ \mathrm{s}^{-1}$, giving pump frequencies $\omega_{1,0}$ of 98.7 and 97.7 in normalized units. The length and width of the computational window containing the laser pulse are $9000c/\omega_p$ and $150c/\omega_p$, respectively, with a corresponding grid resolution of 180×50. The total length of plasma can be up to one "Tajima-Dawson" dephasing length $L_{dp} = 3 \times 10^4 c/\omega_p$.[1] It is necessary to include a sufficiently large number of electromagnetic modes, at least 20 in this case, so that the run time does not exceed the time taken for the outermost sideband intensities to become comparable to the pump intensities. After this time, energy is reflected back towards the pumps, which leads to unphysical solutions for the cascade and plasmon alike.

(i) *Resonant plasmon* ($C + D > 0$).—We consider an example here in which $C > D$ according to expressions (4) and (5). The laser pumps are Gaussian in both directions with intensity $I_0 = 8 \times 10^{14}\ \mathrm{W\,cm^{-2}}$ and normalized quiver velocity $a_0 = 0.0257$. The pulse length is chosen to minimize the initial plasmon wake: $\tau = \tau_{ks}$ where τ_{ks} is the relativistic detuning period defined in

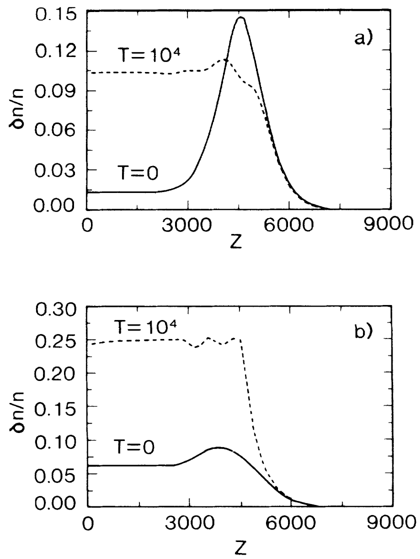

FIG. 1. Normalized plasmon amplitude $e=\delta n/n$ in transformed frame on axis ($x=0$) at times $T=0$ and $T=10^4\omega_p^{-1}$ for (a) resonant case, $\Delta n/n=0$, and (b) detuned case, $\Delta n/n=0.68\%$. The grid resolution is 180 (z) by 50 (x).

Ref. 6. The pulse width is $30c/\omega_p$ (FWHM).

The numerical solution for the plasmon is shown in Fig. 1(a). The amplitude has fallen by 25% in a time $T=10^4\omega_p^{-1}$ and the position of the maximum has slipped backwards. This is the result of a fall in em intensity due to outward spreading from the beam axis. The effective detuning period τ_{ks} is therefore increased, but the average length of the cascade envelope, $\langle\tau\rangle$ is unchanged, so that we now have $\tau<\tau_{ks}$, and the plasmon is not driven as strongly.

(ii) *Detuned plasmon* ($C+D<0$).—The laser parameters are the same as before, but we use a density mismatch $\Delta n/n=0.68\%$. This is chosen to be slightly greater than the optimum of 0.65% given by Eq. (7), so that the initial plasmon wake is reduced. Equation (3) then predicts an initial focusing of the cascade envelope.

Figure 1(b) again shows the plasmon amplitude on axis at $T=0$ and $T=10^4$, the latter time corresponding to a distance $L_{dp}/3$. In contrast to Fig. 1(a), we see that the maximum plasmon amplitude has *increased* by a factor of 3. This is due to focusing of the cascade envelope in the plasmon buildup region as seen in Fig. 2. In Fig. 2(a) the envelope has decreased in width from $30c/\omega_p$ to approximately $12c/\omega_p$. Some small-scale structure has appeared towards the rear of the em envelope but it is apparent from Fig. 2(b) that this has not affected the quality of the plasmon.

After a time $T=10^4$ the phase of the electric field e_z at position $z=4500$, $x=0$ has advanced by nearly $\pi/2$. This implies that the phase fronts will drift relative to a particle injected at this point over a distance $L_{dp}/3$. To avoid premature dephasing, we could increase the density mismatch so that the focusing condition (8) is just satisfied. The focusing rate will then be slower than for the "optimum" configuration given by (7).

There is, however, another consideration which affects our choice of laser parameters. The time scale for ponderomotive depletion of ions from the focal spot is given by $\tau_i\simeq 2^{3/2}(m_i/Zm_e)^{1/2}(\delta n/n)^{-1}\sigma$. For our parameters this is about $6000\omega_p^{-1}$, which is several times the plasmon rise time in Fig. 1. We conclude that the effect may be significant in this case, but is less so at higher intensities where the plasmon time scale is shorter. We must therefore choose the laser intensity and spot size carefully in order to capitalize on the focusing effect.

Physically, "cascade focusing" can be explained by our considering the relative phase velocity of the em

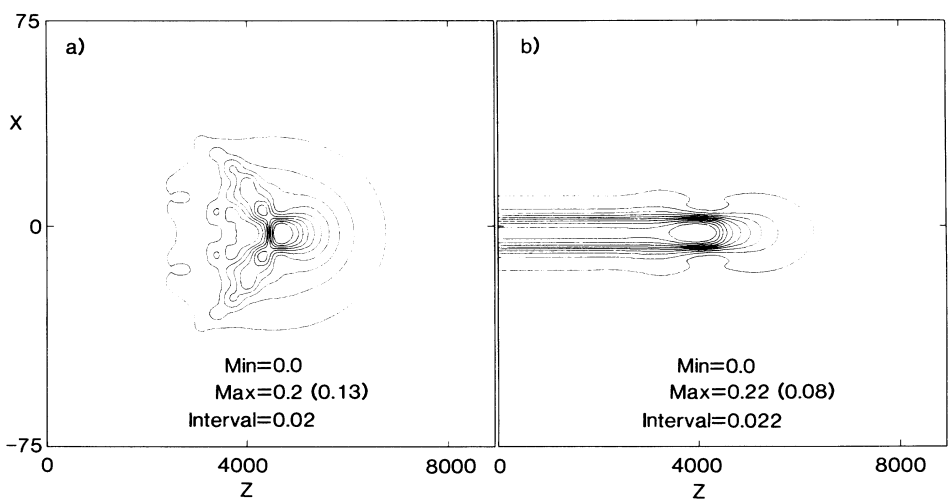

FIG. 2. Contour plots of (a) total normalized em wave action $\sum\omega_m\ |a_m(z,x)|^2$ and (b) plasmon amplitude $|e(z,x)|$ at time $T=10^4\omega_p^{-1}$. The initial maxima are given in parentheses.

waves across the focal spot, in a similar fashion to relativistic self-focusing.[16] One can show that the nonlinear refractive index for a scattered cascade mode depends on the frequency mismatch. If the plasma wave is driven below ω_p, the phase velocity of the scattered wave is lower at the beam center than on either side, and so the phase fronts curve inwards and the wave focuses.

Apart from improving the prospects for beat-wave particle acceleration, the cascade-focusing effect could be used more generally as a means of focusing low-irradiance laser light in plasmas and possibly other nonlinear media. Further study is needed for high irradiances, however, to determine the comparative importance of relativistic self-focusing and the effect of higher-order couplings to plasma-wave harmonics.

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