Ladder Approximation for Spontaneous Chiral-Symmetry Breaking

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(Received 6 July 1988)

This paper examines the validity of the ladder approximation in gauge theories such as technicolor theories in which the coupling is a slowly running function of momentum and is large enough to trigger spontaneous chiral-symmetry breaking. We find that the next-higher-order terms beyond the ladder approximation amount to only a 1%–20% correction, depending on the fermion representation. This indicates that the ladder expansion may provide a much better description of chiral-symmetry breaking than previously thought.

PACS numbers: 12.50.Lr, 11.30.Qc, 11.30.Rd

Technicolor, a new strong gauge interaction of massless fermions, is intended to be the dynamical agent of electroweak-symmetry breaking.[1] At its characteristic energy scale $\Lambda_{TC} \sim 250$ GeV, the technicolor (TC) interaction breaks its fermions' chiral symmetries. When the electroweak symmetry $SU(2)_L \otimes U(1)$ is properly embedded in these symmetries, it is broken down to electromagnetic U(1). To communicate this symmetry breaking to ordinary fermions, and so give them nonvanishing current masses, Dimopoulos and Susskind[2] and Eichten and Lane[3] placed the TC gauge group into a larger one, known as extended technicolor (ETC). This couples the ordinary fermions to technifermions and generates quark and lepton masses m of order

$$m \sim \langle \bar{\Psi}\Psi \rangle_{ETC} / \Lambda_{ETC}^2. \tag{1}$$

Here, $\langle \bar{\Psi}\Psi \rangle_{ETC}$ is the technifermion bilinear condensate renormalized (cut off) at the ETC energy scale Λ_{ETC} at which the breakdown ETC $\rightarrow$ TC$\times \cdots$ takes place.

An estimate of Λ_{ETC} may be obtained from Eq. (1). If, as happens in QCD, the asymptotic freedom of the TC interaction leads to a weak-coupling setting in rapidly above Λ_{TC}, then the anomalous dimension γ_m of $\bar{\Psi}\Psi$ is small and $\langle \bar{\Psi}\Psi \rangle$ suffers only a logarithmic renormalization in scaling from Λ_{ETC} down to Λ_{TC}. Hence,

$$\langle \bar{\Psi}\Psi \rangle_{ETC} = \exp\left[\int_{\Lambda_{TC}}^{\Lambda_{ETC}} \frac{d\mu}{\mu} \gamma_m(\alpha(\mu))\right] \langle \bar{\Psi}\Psi \rangle_{TC} \simeq \langle \bar{\Psi}\Psi \rangle_{TC} \simeq \Lambda_{TC}^3, \tag{2}$$

and the ETC scale required to generate a u- or d-quark mass of 10 MeV is $\Lambda_{ETC} \simeq 40$ TeV. Unfortunately, the ETC interactions also generally induce effective four-fermion flavor-changing neutral-current Lagrangians for quarks and leptons with effective couplings $\sim \theta^2/\Lambda_{ETC}^2$.[3] Here, θ is some presumably not very small mixing angle. The most stringent constraints on these couplings come from $|\Delta S| = 2$ effects in the neutral-kaon system, which require $\Lambda_{ETC}/\theta \gtrsim 500$–$1000$ TeV.

The most promising mechanism for elimination of large flavor-changing neutral-current interactions in ETC models may be traced back to work by Holdom.[4] He suggested that a TC interaction with a nontrivial ultraviolet fixed point could generate a large anomalous-dimension enhancement for $\langle \bar{\Psi}\Psi \rangle_{ETC} / \langle \bar{\Psi}\Psi \rangle_{TC}$, so that a larger Λ_{ETC} would be required to produce a fixed mass m in Eq. (1). Because no realistic theory with a nontrivial ultraviolet fixed point has ever been exhibited, Holdom's work must be regarded as speculative. Recently, however, Appelquist, Karabali, and Wijewardhana have shown how to obtain naturally the desired result of raising Λ_{ETC} without the troublesome assumption of a nontrivial uv fixed point.[5] These authors assumed an asymptotically free TC theory in which the β function is small $[\beta(\alpha)/\alpha \ll 1]$ so that the TC coupling $\alpha(p)$ runs very slowly for a large range of momenta, $\Lambda_{TC} \lesssim p \lesssim \Lambda$, where $\Lambda_{TC} \ll \Lambda < \Lambda_{ETC}$. Near Λ_{TC}, $\alpha(p)$ is expected to be close to a "critical value" α_c which seems to be required for the occurrence of technifermion chiral-symmetry breakdown.[6-9] Then $\alpha(p) \simeq \alpha_c$ over this large momentum range. In the lowest-order computations made so far, $\alpha_c = \pi/3C_2(R)$, where $C_2(R)$ is the quadratic Casimir of the complex technifermion representation R. In this approximation, $\gamma_m[\alpha(p)] \simeq 1 - [1-\alpha(p)/\alpha_c]^{1/2} \simeq 1$, and $\langle \bar{\Psi}\Psi \rangle_{ETC} / \langle \bar{\Psi}\Psi \rangle_{TC} \simeq \langle \bar{\Psi}\Psi \rangle_{\Lambda} / \langle \bar{\Psi}\Psi \rangle_{TC} \simeq \Lambda/\Lambda_{TC}$. From Eq. (1), then, a 10-MeV quark mass may be generated by $\Lambda_{ETC} = 1000$ TeV for Λ as low as 160 TeV.[10] What makes this "walking technicolor" scenario so attractive is that ETC models typically contain a large number of technifermions and so a small TC β function above Λ_{TC} is not unlikely.

The determination of γ_m involves the study of the behavior, for $p \gg \Lambda_{TC}$, of the dynamical mass function

$\Sigma(p)$ in the renormalized technifermion propagator $S(p)=[pA(p)-\Sigma(p)]^{-1}$. The starting point for this analysis is the Schwinger-Dyson gap equation for the zero-bare-mass fermion:

$$\Sigma(p;\mu,\alpha,\xi)=\int\frac{d^4k}{(2\pi)^4}K(p,k;\mu,\alpha,\xi)\frac{\Sigma(k;\mu,\alpha,\xi)}{k^2A^2(k)+\Sigma^2(k)}. \qquad (3)$$

In Eq. (3), the integration is over Euclidean momentum. K is the appropriate projection of the fermion-antifermion scattering kernel; μ is a renormalization scale; $\alpha=\alpha(\mu)$ and $\xi=\xi(\alpha,\mu)$ are the renormalized coupling and covariant gauge parameter, respectively. To determine $\Sigma(p)$ for $p\gg\Lambda_{TC}$, Eq. (3) can be linearized. This follows from the k dependence of the kernel and from the fact that $\Sigma(k)$ turns out to be no larger than Λ_{TC}. For $p\gg\Lambda_{TC}$, the integral is then dominated by momenta k of order p, justifying the linearization to a good approximation [see Ref. 11 for a detailed analysis of this linearization through $O(\alpha^2)$].

For this regime of fairly large momenta, all previous computations have been restricted to the renormalization-group-improved ladder approximation to the kernel in the Landau gauge, $\xi=0$. In this approximation, $A(k)=1$, and the other corrections can be assembled into the running coupling constant of the theory. The integral equation then becomes

$$\Sigma(p)=\frac{3C_2(R)}{4\pi}\int_0^\infty\frac{dk^2}{M^2}\Sigma(k)\alpha(M), \qquad (4)$$

where $M^2=\max(p^2,k^2)$ and $\alpha(M)$ is the running coupling evaluated at the larger of the two momenta.

With use of the above approximation, slowly running theories may be further simplified in the region $\Lambda_{TC}<p<\Lambda$ by our setting $\alpha(M)\simeq\alpha(\Lambda_{TC})$. The equation then has scale-invariant solutions of the form

$$\Sigma(p)=\Sigma(\mu)(\mu^2/p^2)^b, \qquad (5)$$

where

$$b(1-b)=\frac{3\alpha(\Lambda_{TC})C_2(R)}{4\pi}=\frac{\alpha(\Lambda_{TC})}{4\alpha_c}. \qquad (6)$$

Thus,

$$\Sigma(p)=\mu[a_1(\mu^2/p^2)^{b_1}+a_2(\mu^2/p^2)^{b_2}], \qquad (7)$$

with the exponents given by $b_{1,2}=\frac{1}{2}\{1\mp[1-\alpha(\Lambda_{TC})/\alpha_c]^{1/2}\}$. Note that when $\alpha(\Lambda_{TC})$ passes through $\alpha_c\equiv\pi/3C_2(R)$ from below, the two exponents b_1 and b_2 coincide as they change from being real to being complex. It is this transition that is associated with the onset of spontaneous chiral-symmetry breaking.[5,6,8,9,12,13] With the fact that $\alpha(\Lambda_{TC})\simeq\alpha_c$,[5,7-9] it follows that $b_1\simeq b_2\simeq\frac{1}{2}$ and therefore that $\gamma_m\equiv 2b_1\simeq 1$.[14] This is the origin of the enhancement of $\langle\bar{\Psi}\Psi\rangle$.

What is worrisome in this analysis is that both the critical coupling and γ_m are determined by the first term in the expansion whose natural parameter $\alpha C_2(R)/\pi$ could well be of order unity. How can we be sure that this coincidence of the onset of chiral-symmetry breaking with $\gamma_m=1$ and condensate enhancement is not just an artifact of the ladder approximation? Accurate strong-coupling computations would certainly help to address this question. An alternative approach is to compute the higher-order corrections to the kernel in Eq. (3) and see how big they are when α is large enough to trigger chiral-symmetry breaking. We have done this through second order and we have found that the corrections are fairly small. We interpret this as evidence that a perturbative expansion of the kernel may be a more useful approach to spontaneous chiral-symmetry breaking than previously thought.

There is good reason to believe that in second order and higher, just as in lowest order, the determination of the critical coupling and the computation of γ_m can be restricted to the linearized version of Eq. (3). That γ_m is determined by the linearized equation is, of course, to be expected. Although it is not so obvious that the critical coupling can likewise be determined, the essential features of the lowest-order analysis[5-12] described above indicate this to be the case. There, the nonlinear regime $k\le\Sigma(k)$ is important to set the overall scale of the dynamical mass function, but in slowly running theories[5] and in nonrunning models,[6,8,9,12,13] the critical coupling strength and behavior near criticality are determined by the linearized gap equation. These analyses also suggest that the linearized equation will remain adequate in higher orders.

In this paper, we shall assume this to be the case. Working in a limit in which the gauge coupling does not run at all, we compute the linearized kernel in Eq. (3) through $O(\alpha^2)$ in an arbitrary ξ gauge.[11] This then becomes a computation of the higher-order corrections to $b(1-b)$ [Eq. (6)]. The critical coupling will then be determined, as above, by the condition $b(1-b)=\frac{1}{4}$, but with the higher-order corrections included. The gap equation to the relevant order includes the various vertex and self-energy corrections as well as the crossed ladder contribution to the kernel. We assume a non-Abelian gauge group G with n technifermion flavors, each belonging to the single complex representation R. The resulting expression is very complicated and is being evaluated numerically.[11,15] For the purposes of this paper, however, we consider an approximation to the kernel in which we keep only those terms that survive in the limit

$$m^2/M^2=\min(k^2,p^2)/\max(k^2,p^2)\to 0.$$

This approximation retains large logarithms (which involve M^2/μ^2 only) and constants. This is the same approximation used in the previous analysis of walking technicolor[5] except that we have improved the kernel by one power of α and we do not restrict our discussion to

Landau gauge. While we believe that this approximation is adequate to compute the next-order corrections to Eq. (6), ongoing analytical and numerical work is designed to settle this question.

The computation from the above approximations leads to the following integral equation for $\Sigma(p)/A(p)$:

$$\Sigma(p)/A(p)=\int\frac{dk^2}{M^2}\frac{\Sigma(k)}{A(k)}\frac{A(M)}{A(m)}\{\kappa_1[\xi(M)]\alpha(M)+\kappa_2[\xi(M)]\alpha^2(M)\}+O(\alpha^3), \tag{8}$$

where, in the limit $\Sigma=0$, $A(M)$ is given by the solution to the renormalization-group equation,

$$A(M)=A(\mu)\exp\left[-\int_\mu^M\frac{d\mu'}{\mu'}\gamma_2(\alpha(\mu'),\xi(\mu'))\right]. \tag{9}$$

γ_2 is twice the anomalous dimension of the fermion field and is given through $O(\alpha^2)$ by[16]

$$\gamma_2=\frac{\alpha\xi}{2\pi}C_2(R)+\tfrac{1}{2}(\alpha/4\pi)^2C_2(R)[(25+8\xi+\xi^2)C_2(G)-8nT(R)-6C_2(R)]. \tag{10}$$

In Eq. (10), $C_2(G)$ is the quadratic Casimir of the adjoint representation of the group G and $T(R)$ is the trace of the square of the fermion representation matrices [normalized to $\frac{1}{2}$ for the fundamental representation of SU(N)].

The expansion coefficients in Eq. (8) are found to be

$$\kappa_1(\xi)=(3+\xi)C_2(R)/4\pi,$$
$$\kappa_2(\xi)=[C_2(R)/(4\pi)^2][\tfrac{1}{12}(313+42\xi+9\xi^2)C_2(G)-\tfrac{20}{3}nT(R)-(3+\xi)^2C_2(R)]. \tag{11}$$

The running coupling and gauge parameters in Eq. (8) are solutions of $d\alpha(t)/dt=\beta(\alpha(t))$, $d\xi(t)/dt=\beta_G(\alpha(t),\xi(t))$, where $t=\ln(M^2/\mu^2)$, and to lowest order[16]

$$\beta(\alpha)=-(\alpha^2/12\pi)[11C_2(G)-4nT(R)],$$
$$\beta_G(\alpha,\xi)=(\xi\alpha/24\pi)[(13-3\xi)C_2(G)-8nT(R)].$$

In general, the functions Σ and A have scale-invariant forms

$$\Sigma(p)=\Sigma(\mu)(\mu^2/p^2)^{b+\lambda},\quad A(p)=A(\mu)(\mu^2/p^2)^\lambda, \tag{12}$$

if and only if neither $\alpha(M)$ nor $\xi(M)$ run with M. To the order α^2 we have calculated, the former is guaranteed by the nonrunning choice $11C_2(G)=4nT(R)$, and the latter can then be insured by the gauge choice[17] $\xi=\xi^*=0$ or -3. If we adopt both these conditions, the scale-invariant *Ansatz* (12) can be used in Eq. (8) to give

$$b(1-b)=-\lambda(1+\lambda)+[\kappa_1(\xi^*)\alpha+\kappa_2(\xi^*)\alpha^2](1+2\lambda)+O(\alpha^3). \tag{13}$$

Using $\lambda=\frac{1}{2}\gamma_2(\xi^*)$, Eq. (13) becomes

$$b(1-b)=\tfrac{1}{4}(\alpha/\alpha_c+(\alpha/\alpha_c)^2\{[14+\xi^*(3+\xi^*)]C_2(G)-15C_2(R)\}/72C_2(R))+O((\alpha/\alpha_c)^3). \tag{14}$$

This equation is our main result. As expected, $b(1-b)$ is independent of the gauge parameter with either $\xi^*=0$ or -3.[18] Note in particular that this occurs despite the vanishing of κ_1 for $\xi=-3$. Furthermore, Eq. (14) agrees with the gauge-invariant quantity $(\gamma_m/2)(1-\gamma_m/2)$, providing that the expression for $\gamma_m/2$,[16]

$$\tfrac{1}{2}\gamma_m=\tfrac{1}{4}\{\alpha/\alpha_c+(\alpha/\alpha_c)^2[97C_2(G)-20nT(R)+9C_2(R)]/216C_2(R)\}+O((\alpha/\alpha_c)^3) \tag{15}$$

is evaluated using the condition $11C_2(G)=4nT(R)$.

In Eq. (14), the coefficient of $(\alpha/\alpha_c)^2$,

$$\chi=[14C_2(G)-15C_2(R)]/72C_2(R), \tag{16}$$

is generally quite small. This is partly due to the large denominator factor of 72, a consequence of normalizing α to α_c. Let us make the popular choice of TC group, $G=$SU(N), with $C_2(G)=N$. Then χ is small also because of an approximate cancellation between the two numerator factors. For fermions in the fundamental representation of SU(N), this may be partly traced to the fact that nonplanar graphs such as the crossed ladder are suppressed by $1/N^2$. For such fermions, $C_2(R)=(N^2-1)/2N$, and $\chi\simeq0.2$ for all $N\geq3$. The cancellation is even better (perhaps fortuitously) for fermions in the second-rank tensor representations, with $C_2(R)=(N\pm2)(N\mp1)/N$. For the antisymmetric tensor, $|\chi|\lesssim0.1$ for $N\geq4$; for the symmetric tensor, $|\chi|\lesssim0.03$ for $N\geq4$. For very large representations, χ approaches $-\frac{5}{24}$, the value it would have in an Abelian gauge theory with the β function artificially set equal to zero.[6,12]

Because of the smallness of χ, the second-order term in Eq. (14) will be small when α is of order α_c, the lowest-order critical coupling strength required to trigger chiral-symmetry breaking. Conversely, with the second-order critical coupling α_{c2} defined by the condition $b(1-b)=\frac{1}{4}$, our $O(\alpha^2)$ calculations have shown that

$$\alpha_{c2}/\alpha_c = \frac{1}{2\chi}[(1+4\chi)^{1/2}-1] \simeq 1-\chi \tag{17}$$

is always close to 1. Thus the second-order evidence is that the conclusions drawn from the lowest-order analysis of chiral-symmetry breaking in slowly running gauge theories are both qualitatively and quantitatively accurate. In particular, the important feature of condensate enhancement, leading in technicolor theories to a larger value of the ETC scale for fixed fermion masses, is not substantially changed by the second-order corrections.

There is some cause to believe that our conclusions will not be changed significantly by terms of order α^3 and higher. For example, in each order cancellations can be expected between terms with an even and an odd number of fermion loops. To check convergence explicitly, both γ_2 and the kernel K would have to be computed to next order. Only that part of the kernel computation that corresponds to the two-loop β function[19] has already been done.

We thank Andy Cohen, Howard Georgi, Daniel Nash, Tatsu Takeuchi, and L. C. R. Wijewardhana for helpful discussions. The hospitality of the Aspen Center for Physics, where this work was begun, is gratefully acknowledged. One of us (K.L.) also acknowledges the hospitality of the Harvard University Physics Department where some of this work was carried out. This work was supported in part by Yale University under Department of Energy Contract No. DE-AC02-76ER03075 (T.A.), by Boston University under Department of Energy Contract No. DE-AC02-86ER40284 (K.L.), and by Ohio State University under Department of Energy Contract No. DE-AC02-76ER01545 (K.L. and U.M.).

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[18] In quenched QED, b is in general independent of ξ.

[19] It is not difficult to find slowly running theories for which the β function is reasonably convergent through two loops (see Ref. 5, third paper).