Dipole-Quadrupole Interaction in Spherical Nuclei and Berry Potentials

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We show that SO(3) gauge potentials arise in the Born-Oppenheimer description of the interaction between the dipole and quadrupole vibrations in even-even spherical nuclei and derive the effective Hamiltonian for the quadrupole motion.

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It has recently been appreciated that gauge potentials, the Berry connections, can arise in the adiabatic description of physical situations involving two different time scales.[1] Such a situation can occur in spherical nuclei where the frequency of the quadrupole oscillations is typically much smaller than that of the dipole excitations. The coupling between these two modes is one of the mechanisms responsible for the broadening of the giant-dipole line,[2,3] and, in view of the two time scales involved, a Born-Oppenheimer type of approximation seems best suited to analyze this effect. The purpose of this Letter is to show that Berry connections do indeed occur in such an approach.

We shall describe the fast dipole vibrations in terms of three collective coordinates $\alpha_{1\mu}$ specifying the relative distance between the centers of mass of the neutrons and the protons, and the slow quadrupole vibrations in terms of the five tensor component $\alpha_{2\mu}$ introduced in the multiple expansion of the nuclear surface.[3] The Hamiltonian will be of the form

$$H=H_1(\alpha_1)+H_2(\alpha_2)+H_{12}(\alpha_1,\alpha_2). \tag{1}$$

In the harmonic approximation,

$$H_\lambda(\alpha_\lambda)=\frac{1}{2B_\lambda}\sum_\mu \pi^*_{\lambda\mu}\pi_{\lambda\mu}+\frac{C_\lambda}{2}\sum_\mu \alpha^*_{\lambda\mu}\alpha_{\lambda\mu},\quad \lambda=1,2, \tag{2}$$

where $\pi_{\lambda\mu}$ is the momentum conjugate to $\alpha_{\lambda\mu}$, while B_λ and C_λ are, respectively, the inertia and restoring force parameters. The interaction term H_{12} is uniquely specified from invariance considerations if one limits oneself to cubic terms containing no time derivatives:

$$H_{12}=\frac{K}{\sqrt{5}}\sum_{\mu\nu}(-1)^\mu\langle 11\nu\mu-\nu\,|\,2\mu\rangle\alpha_{1\nu}\alpha_{1\mu-\nu}\alpha_{2-\mu}. \tag{3}$$

It is convenient to replace, as usual,[3] the five $\alpha_{2\mu}$'s by two deformation parameters, β and γ, that specify the shape of the ellipsoid in the frame coinciding with the principal axes, and three Euler angles (ϕ,θ,ψ) that give the orientation of the intrinsic frame with respect to the laboratory,[4]

$$\alpha_{2\mu}=\sum_\nu D^2_{\mu\nu}(\phi,\theta,\psi)\bar{\alpha}_{2\nu}, \tag{4}$$

with

$$\bar{\alpha}_{20}=\beta\cos\gamma,\quad \bar{\alpha}_{22}=\bar{\alpha}_{2-2}=(\beta/\sqrt{2})\sin\gamma,$$
$$\bar{\alpha}_{21}=\bar{\alpha}_{2-1}=0. \tag{5}$$

The fast Hamiltonian $H_F(\alpha_1,\alpha_2)\equiv H_1(\alpha_1)+H_{12}(\alpha_1,\alpha_2)$ can then be written in the form

$$H_F(\alpha_1,\alpha_2)=U(\phi,\theta,\psi)H_F(\alpha_1,\beta,\gamma)U^\dagger(\phi,\theta,\psi), \tag{6}$$

where

$$U(\phi,\theta,\psi)=e^{-i\phi J_z^{(1)}/\hbar}e^{-i\theta J_y^{(1)}/\hbar}e^{-i\psi J_z^{(1)}/\hbar} \tag{7}$$

is the rotation operator that acts on the dipole variables only, while $H_F(\alpha_1,\beta,\gamma)$ describes an anisotropic harmonic oscillator, the principal axes of which coincide with those of the laboratory. In a Cartesian basis,

$$H_F(\alpha_1,\beta,\gamma)=\sum_{i=1}^{3}\left(\frac{1}{2B_1}\pi_i^2+\frac{1}{2}K_i\alpha_{1i}^2\right), \tag{8}$$

with

$$K_i(\beta,\gamma)=C_1\left[1+\frac{4K\beta}{\sqrt{30}C_1}\cos\left(\gamma-\frac{2\pi i}{3}\right)\right]. \tag{9}$$

If we denote $|\mathbf{n};\beta,\gamma\rangle$ by the eigenstates of $H_F(\alpha_1,\beta,\gamma)$, those of $H_F(\alpha_1,\alpha_2)$ will be given by $U(\phi,\theta,\psi)|\mathbf{n};\beta,\gamma\rangle$.

We shall be interested in states $|1_i;\beta,\gamma\rangle$ having one dipole phonon along one of the three principal axes $(i=1,2,3)$. They are generated by our letting the creation operators

$$a_i^\dagger(\beta,\gamma)=\left(\frac{B_1K_i}{4\hbar^2}\right)^{1/4}\alpha_{1i}-i\frac{\pi_{1i}}{(4\hbar^2B_1K_i)^{1/4}}, \tag{10}$$

act on the deformed ground state $|0;\beta,\gamma\rangle$, which is defined through the condition $a_i(\beta,\gamma)|0;\beta,\gamma\rangle=0$, and they have the energies

$$\epsilon_i(\beta,\gamma)=\frac{5}{2}\hbar\left(\frac{K_i(\beta,\gamma)}{B_1}\right)^{1/2}. \tag{11}$$

Since these three states are quasidegenerate, a thorough mixing should occur under the slow motion. We can thus expect that a non-Abelian Berry connection will show up in the effective Hamiltonian for the quadrupole oscillations.

We shall seek eigenfunctions of the total Hamiltonian (1) with the following form

$$\Psi(\alpha_1;\beta,\gamma,\phi,\theta,\psi) = \sum_{i=1}^{3} \psi_i(\beta,\gamma,\phi,\theta,\psi) U(\phi,\theta,\psi) \,|\, 1_i;\beta,\gamma\rangle. \quad (12)$$

Because of the rotational invariance of H, the total angular momentum

$$\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)} \quad (13)$$

is conserved. Moreover, since the fast eigenfunctions $U(\phi,\theta,\psi)\,|\,1_i;\beta,\gamma\rangle$ are invariant under combined rotations of the dipole and quadrupole variables, the action of $\mathbf{J}$ on Ψ reduces to that of $\mathbf{J}^{(2)}$ on the $\psi_i(\alpha_2)$'s. A partial separation of the variables will thus result from diagonalization of $(\mathbf{J}^{(2)})^2$ and $J_z^{(2)}$, whence, with $\hbar^2 J(J+1)$ and $\hbar M$ representing the respective eigenvalues, one obtains

$$\psi_{iJM}(\beta,\gamma,\phi,\theta,\psi) = \left(\frac{2J+1}{8\pi^2}\right)^{1/2} \sum_{N=-J}^{J} F_{iN}^{J}(\beta,\gamma) D_{MN}^{J}(\phi,\theta,\psi). \quad (14)$$

Since the giant-dipole resonance in spherical even-even nuclei is excited by absorption of $E1$ photons on a $J=0$ ground state, we shall concentrate on $J=1$ states. It is extremely important that the total wave functions, given by (12) and (14), should have the right symmetry properties. There are 24 ways of defining the coordinates$(\beta,\gamma,\phi,\theta,\psi)$, each of which corresponds to a different way of attaching a right-handed frame to the principal axes, and it must be ensured that the total wave function is invariant under the 24 transformations that merely relabel and reorient the intrinsic axes. It has been shown[5] that these are generated by three operators R_1, R_2, and R_3. If we denote by ξ_1, ξ_2, and ξ_3 the principal axes with respect to which the coordinates $(\beta,\gamma,\phi,\theta,\psi)$ have been defined, R_1 corresponds to the reversal of the ξ_2 and ξ_3 axes, R_2 is a rotation of $\pi/2$ about the ξ_3 axis, and R_3 is a circular permutation of the intrinsic axes: $(\xi_1, \xi_2, \xi_3) \rightarrow (\xi_3, \xi_1, \xi_2)$. It therefore suffices to enforce the stability of Ψ_{1M} under these three transformations. For $J=1$, the following constraints on the expansion coefficients F_{iN} result from imposing the invariance Ψ_{1M} under R_1 and R_2:

$$F_{10}(\beta,\gamma) = F_{20}(\beta,\gamma) = F_{31}(\beta,\gamma) = F_{3-1}(\beta,\gamma) = 0, \quad (15a)$$

$$F_{1-1}(\beta,\gamma) = -F_{11}(\beta,\gamma), \quad (15b)$$

$$F_{21}(\beta,\gamma) = F_{2-1}(\beta,\gamma) = iF_{11}(\beta,-\gamma), \quad (15c)$$

$$F_{30}(\beta,\gamma) = F_{30}(\beta,-\gamma). \quad (15d)$$

We are thus left with only two functions, namely $F_{11}(\beta,\gamma)$ and $F_{30}(\beta,\gamma)$, and the complete wave function reduces to

$$\Psi_{1M}(\alpha_1;\beta,\gamma,\phi,\theta,\psi) = \left(\frac{3}{8\pi^2}\right)^{1/2} \{F_{11}(\beta,\gamma)[D_{M1}^{1}(\phi,\theta,\psi) - D_{M-1}^{1}(\phi,\theta,\psi)]U(\phi,\theta,\psi)\,|\,1_1;\beta,\gamma\rangle$$
$$+ iF_{11}(\beta,-\gamma)[D_{M1}^{1}(\phi,\theta,\psi) + D_{M-1}^{1}(\phi,\theta,\psi)]U(\phi,\theta,\psi)\,|\,1_2;\beta,\gamma\rangle$$
$$+ F_{30}(\beta,\gamma) D_{M0}^{1}(\phi,\theta,\psi) U(\phi,\theta,\psi)\,|\,1_3;\beta,\gamma\rangle\}. \quad (16)$$

Finally, invariance under R_3 implies the relations

$$F_{11}(\beta,-\gamma) = -(1/\sqrt{2}) F_{30}(\beta,\gamma+2\pi/3), \quad (17a)$$

$$F_{11}(\beta,\gamma+2\pi/3) = -(1/\sqrt{2}) F_{30}(\beta,\gamma), \quad (17b)$$

which, together with (15d), completely specify $F_{11}(\beta,\gamma)$ and $F_{30}(\beta,\gamma)$ in the entire (β,γ) domain once they are known for $-\pi/3 \le \gamma \le \pi/3$.

In terms of $(\beta,\gamma,\phi,\theta,\psi)$, the quadrupole Hamiltonian naturally splits into a vibrational and a rotational part, $H_2 = H^{\text{vib}} + H^{\text{rot}}$, with

$$H^{\text{vib}} = T^{\text{vib}} + \frac{C_2}{2}\beta^2 = -\frac{\hbar^2}{2B_2}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right) + \frac{C_2}{2}\beta^2, \quad (18)$$

and

$$H^{\text{rot}} = \frac{\hbar^2}{2}\sum_{l=1}^{3}\frac{(J_l^{(2)})^2}{\mathcal{J}_l}, \quad (19)$$

where $\mathcal{J}_l = 4B_2\beta^2\sin^2(\gamma - 2\pi l/3)$ and $J_l^{(2)}$ is the component of the quadrupole angular momentum operator along the intrinsic axis ξ_l.

The coupled Born-Oppenheimer (BO) equations for the slow motion,

$$\sum_{i=1}^{3} H_{ji}^{\text{BO}}\psi_i = E\psi_j, \quad (20)$$

are obtained by projecting the complete eigenvalue equation, $H\Psi = E\Psi$, on $U(\phi,\theta,\psi)\,|\,1_j;\beta,\gamma\rangle$:

$$H^{\mathrm{BO}}_{ji}\psi_i(\beta,\gamma,\phi,\theta,\psi) = \langle 1_j;\beta,\gamma\,|\,U^\dagger(\phi,\theta,\psi)(H_2+H_F)\psi_i(\beta,\gamma,\phi,\theta,\psi)U(\phi,\theta,\psi)\,|\,1_i;\beta,\gamma\rangle. \tag{21}$$

This is the appropriate point for introducing the relevant Berry connection, which is defined as

$$A_{ji} = \langle 1_j;\beta,\gamma\,|\,U^\dagger(\phi,\theta,\psi)dU(\phi,\theta,\psi)\,|\,1_i;\beta,\gamma\rangle. \tag{22}$$

The exterior derivative d that acts on the slow variables can be decomposed as

$$d = d\beta\frac{\partial}{\partial\beta} + d\gamma\frac{\partial}{\partial\gamma} + \sum_{l=1}^{3}\omega_l R_l, \tag{23}$$

where the ω_l's are the Maurer-Cartan forms dual to the vector fields $R_l[\omega_l(R_m)=\delta_{lm}]$. Since the eigenstates $|\,1_i;\beta,\gamma\rangle$ are real, the β and γ components of A_{ji} vanish. Moreover, since $U^\dagger R_l U = -J_l^{(1)}/\hbar$, one has

$$A_{ji} = \sum_{l=1}^{3}\omega_l (A_l)_{ji}, \tag{24}$$

with

$$(A_l)_{ji} = -\langle 1_j;\beta,\gamma\,|\,J_l^{(1)}/\hbar\,|\,1_i;\beta,\gamma\rangle. \tag{25}$$

In accordance with general results,[6] A is seen to be invariant under rotations of the quadrupole variables. A little algebra now shows that

$$H^{\mathrm{BO}}_{ji} = \delta_{ji}T^{\mathrm{vib}} + \frac{\hbar^2}{2}\sum_l \frac{(R_l+A_l)^2_{ji}}{\mathcal{J}_l} + V_{ji}(\beta,\gamma), \tag{26}$$

with

$$V_{ji}(\beta,\gamma) = \left[\epsilon_j(\beta,\gamma) + \frac{C_2}{2}\beta^2\right]\delta_{ji} - \frac{\hbar^2}{2B_2}\langle 1_j;\beta,\gamma\,|\,\frac{\partial^2}{\partial\beta^2} + \frac{1}{\beta^2}\frac{\partial^2}{\partial\gamma^2}\,|\,1_i;\beta,\gamma\rangle + \frac{\hbar^2}{2}\sum_l \frac{\langle 1_j;\beta,\gamma\,|\,(J_l^{(1)}/\hbar)^2\,|\,1_i;\beta,\gamma\rangle - (A_l^2)_{ji}}{\mathcal{J}_l}. \tag{27}$$

In order to compute explicitly $(A_l)_{ji}$ and the matrix elements of $(J_l^{(1)})^2$, it is convenient to express the rotation generators for the dipole variables in terms of the creation and annihilation operators (10),

$$J_l^{(1)} = -i\hbar[p_l(a_m^\dagger a_n - a_m a_n^\dagger) + q_l(a_m a_n - a_m^\dagger a_n^\dagger)], \tag{28}$$

where

$$p_l(\beta,\gamma) = \frac{1}{2}\left[\left(\frac{K_m}{K_n}\right)^{1/4} + \left(\frac{K_n}{K_m}\right)^{1/4}\right],\qquad q_l(\beta,\gamma) = \frac{1}{2}\left[\left(\frac{K_n}{K_m}\right)^{1/4} - \left(\frac{K_m}{K_n}\right)^{1/4}\right], \tag{29}$$

and (l,m,n) is a cyclic permutation of (1, 2, 3). A standard calculation yields

$$\langle 1_j;\beta,\gamma\,|\,J_l^{(1)}/\hbar\,|\,1_i;\beta,\gamma\rangle = p_l(T_l)_{ji}, \tag{30a}$$

$$\langle 1_j;\beta,\gamma\,|\,(J_l^{(1)}/\hbar)^2\,|\,1_i;\beta,\gamma\rangle = p_l^2(T_l^2)_{ji} + q_l^2(1+T_l^2)_{ji}, \tag{30b}$$

where the matrices

$$(T_l)_{ji} = -i(\delta_{jm}\delta_{in} - \delta_{jn}\delta_{im}) \tag{31}$$

are the generators of SO(3). It is equally straightforward to show that

$$\langle 1_j;\beta,\gamma\,|\,\frac{\partial^2}{\partial\beta^2} + \frac{1}{\beta^2}\frac{\partial^2}{\partial\gamma^2}\,|\,1_i;\beta,\gamma\rangle = -\frac{K^2}{60}\delta_{ij}\left[\frac{3}{K_j^2} + \frac{1}{K_m^2} + \frac{1}{K_n^2}\right], \tag{32}$$

where (j,m,n) is a permutation of (1,2,3).

A complete reduction of the slow-motion equations is achieved upon collecting all these results and inserting in (20) the explicit expressions for ψ_i that one reads from (16). After elimination of the angular variables, one is left with two dynamical equations for $F_{11}(\beta,\gamma)$ and $F_{30}(\beta,\gamma)$ that are to be solved for $-\pi/3 \le \gamma \le \pi/3$. The first one reads

$$\left[H^{\mathrm{vib}} + \epsilon_1(\beta,\gamma) + \frac{\hbar^2K^2}{120}\left[\frac{3}{K_1^2} + \frac{1}{K_2^2} + \frac{1}{K_3^2}\right] + \frac{\hbar^2}{2}\left[\frac{q_1^2}{\mathcal{J}_1} + \frac{2p_2^2+q_2^2}{\mathcal{J}_2} + \frac{2p_3^2+q_3^2}{\mathcal{J}_3}\right]\right]F_{11}(\beta,\gamma) - \hbar^2\frac{p_3}{\mathcal{J}_3}F_{11}(\beta,-\gamma) + \frac{\hbar^2}{\sqrt{2}}\frac{p_2}{\mathcal{J}_2}F_{30}(\beta,\gamma) = EF_{11}(\beta,\gamma). \tag{33}$$

The second one is obtained from the latter by first replacing γ by $\gamma-2\pi/3$, and then coming back to γ through Eqs. (17) and the symmetry properties of the various functions of $\gamma(K_l, \mathcal{J}_l$, and so on). We will report elsewhere[7] on the numerical solution of these equations.

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