

Random Multiplicative Processes and Transport in Structures with Correlated Spatial Disorder

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(Received 3 June 1988)

We show that strong spatial correlations in a random resistor network can dramatically alter its transport properties. We calculate the average logarithmic resistance of a topologically one-dimensional model characterized by a random multiplicative process. We find a transport exponent that depends explicitly on the form of the spatial correlations; we also find that this problem is related to *diffusion* in the presence of correlated random fields.

PACS numbers: 05.40.+j, 05.60.+w

The effect of spatial disorder on the properties of materials has become an active area of recent investigation.^{1,2} Spatial disorder is generally taken to be random, for example, by introduction of random bias fields which alternate from point to point in the system. To study physical properties such as transport, most previous work has been based on variations of the classic percolation model in which the disordered material is treated as an uncorrelated network of random bonds (e.g., resistors) that are either open or blocked (finite or infinite resistivity).^{3,4} Thus the spatial disorder is assumed to be completely uncorrelated. However, in many real disordered materials, such as polymers, porous materials, and amorphous systems,¹⁻⁴ the spatial disorder is correlated. For example, if we model the permeability of a porous rock by an array of resistors whose resistances are chosen randomly, then it is possible to find huge resistances neighboring tiny resistances. Such configurations cannot occur in nature since the permeability of a "crack," while random, cannot fluctuate arbitrarily. The spatial disorder is *correlated*.

Here we introduce a topologically one-dimensional model that encompasses the essential physics of *correlated* spatial disorder but is simple enough to be treated analytically.⁵ Consider a set of N resistors in series, where the resistance R_j of resistor j changes in a *correlated* fashion,

$$R_{j+1} \equiv (1 + \epsilon)^{\tau_j} R_j. \quad (1a)$$

Here $\epsilon > 0$ is arbitrary, and τ_j is chosen randomly to be $+1$ or -1 (see Fig. 1). Because neighboring resistors may only differ by a factor of $(1 + \epsilon)$, this model ensures a smooth spatial variation of the resistance. From Eq. (1a), the resistance of bond l is

$$R_l = R_1 (1 + \epsilon)^{\sum_{j=1}^{l-1} \tau_j}. \quad (1b)$$

Note that $\{\tau_j\}$ can be viewed as generating a *walk*; $\tau_j = +1$ corresponds to a step to the right and $\tau_j = -1$

corresponds to a step to the left. We consider two cases: (i) $\{\tau_j\}$ uncorrelated and (ii) $\{\tau_j\}$ with long-range spatial order.

We will be interested in the result of a typical measurement of $R_{\text{tot}} \equiv R_{\text{tot}}(N)$, the total resistance of the N -resistor chain. Our model represents a random multiplicative process, in contrast to the familiar random-walk model, which is a random additive process. In a random walk, the mean-square displacement, for example, coincides with the most probable value. In the present model, and in random multiplicative processes in general, it is natural to find quantities whose mean and most probable value differ markedly.⁶

If we consider the resistance R_{tot} of the entire chain, we find that its average is dominated by improbable configurations of the τ_j (e.g., $\tau_j = 1$ for all j), for which the value of the resistance is large. For case (i), a direct calculation yields $\langle R_l \rangle = \left\{ \frac{1}{2} [(1 + \epsilon) + (1 + \epsilon)^{-1}] \right\}^{l-1}$, resulting in $\ln \langle R_{\text{tot}} \rangle \sim N$.

The *typical* measured value of R_{tot} is dominated by its most probable value; the probability of obtaining for R_{tot} a value that differs by an order of magnitude from the most probable value, vanishes with N . The *typical* value of a single resistor, R_{typ} , is presented by the logarithmic average,⁷

$$R_{\text{typ}} \sim \exp[\langle \ln R_N \rangle] \sim (1 + \epsilon)^{X(N)}. \quad (2)$$

Here $X(N)$ represents the rms displacement in the N -step walk defined by the $\{\tau_j\}$,

$$X(N) \equiv \left\langle \left(\sum_{j=1}^N \tau_j \right)^2 \right\rangle^{1/2}. \quad (3)$$

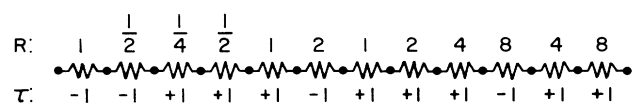


FIG. 1. A realization of the present model with $\epsilon = 1$.

The number of such typical resistances (i.e., typical walks) is of the order of N . Hence from (2) the logarithmic average $\langle \ln R_{\text{tot}} \rangle$ scales as

$$\langle \ln R_{\text{tot}} \rangle \sim \ln(NR_{\text{typ}}) \sim X(N)\ln(1+\epsilon) + \ln N, \quad (4)$$

where the quantity $X(N)$ will depend on the details of the correlations of the τ_j to be determined below. The quantity $\langle \ln R_{\text{tot}} \rangle$ is sensitive to the details of the ensemble, while the quantity $\langle R_{\text{tot}} \rangle$ is simply dominated by one configuration in the ensemble.

In order to calculate $X(N)$, we must define the distribution of the $\{\tau_j\}$. For case (i), where the $\{\tau_j\}$ are uncorrelated, $X(N) = \sqrt{N}$, which is the familiar random-walk result. For case (ii), where the $\{\tau_j\}$ have long-range correlations, $X(N)$ depends on the details of the correlation. We assume that our large but finite system is part of a much larger periodic system of size $\Omega \gg N$. We further assume the distribution to be symmetric under reversal of all the τ_j and that the Fourier transforms of the $\{\tau_j\}$, given by

$$\tau_q \equiv \frac{1}{\sqrt{\Omega}} \sum_{l=1}^{\Omega} \tau_l e^{-iq l}, \quad (5)$$

are correlated through the power-law relation

$$\langle \tau_q \tau_{-q} \rangle \sim 1/q^\lambda, \quad (6)$$

for small q . If two neighboring τ_j tend to be of the same sign (which we call the *ferro* case), then $\lambda > 0$, while if two neighboring τ_j tend to be of opposite sign (the *antiferro* case), then $\lambda < 0$. For uncorrelated $\{\tau_j\}$, we have $\lambda = 0$.

It is straightforward to verify that

$$[X(N)]^2 = \frac{1}{\Omega} \sum_q \langle \tau_q \tau_{-q} \rangle |f(q, N)|^2, \quad (7)$$

where

$$f(q, N) \equiv (e^{-iq(N+1)} - 1)/(e^{-iq} - 1). \quad (8)$$

When $\Omega \rightarrow \infty$ we find, on substituting (6) into (7) and converting the sum to an integral, that the dominant contribution scales for large N as

$$[X(N)]^2 \sim N^{1+\lambda} \quad [\lambda > -1]. \quad (9)$$

Combining (9), (3), and (4) we find

$$\langle \ln R_{\text{tot}} \rangle \sim N^{(1+\lambda)/2} + \ln N \quad [\lambda > -1]. \quad (10)$$

We have confirmed the validity of (10) numerically. To this end, we had to generate a set of $\{\tau_j\}$ that have long-range correlations as in (6). Now each τ_j configuration is in 1:1 correspondence with a N -step random walk: $[X(N)]^2$ is the mean-square displacement, characterized by the fractal dimension d_w , $[X(N)]^2 \sim N^{2/d_w}$. From (9), $d_w = 2/(1+\lambda)$. Therefore we can generate a distribution with any desired λ by generating the corresponding walk with $1 < d_w < \infty$. We

consider the full chain of N sites to consist of strings, each of m sites, where all τ_j in one string have either the same signs (ferro case) or alternating signs (antiferro case). The length m of each string is chosen from the power-law ("Levy flight") distribution

$$P(m) \sim m^{-\beta}. \quad (11)$$

The exponent β determines the correlation parameter λ and therefore d_w . We find that in the ferro case ($\lambda > 0$)

$$d_w = \frac{2}{1+\lambda} = \begin{cases} 1, & \beta \leq 2, \\ 2/(4-\beta), & 2 \leq \beta \leq 3, \\ 2, & \beta \geq 3, \end{cases} \quad (12)$$

while in the antiferro case ($\lambda < 0$)

$$d_w = \frac{2}{1+\lambda} = \begin{cases} 2/(\beta-1), & 1 \leq \beta \leq 2, \\ 2, & \beta \geq 2. \end{cases} \quad (13)$$

Equations (12) and (13) are obtained by one's noting that $\langle \tau_q \tau_{-q} \rangle$ is the Fourier transform of $\langle \tau_0 \tau_l \rangle$ and this is related to $P(m)$ as defined by (11).

Computer simulations agree with our predictions. Figure 2 is a double logarithmic plot of $\langle \ln R_{\text{tot}} \rangle^2$ and the

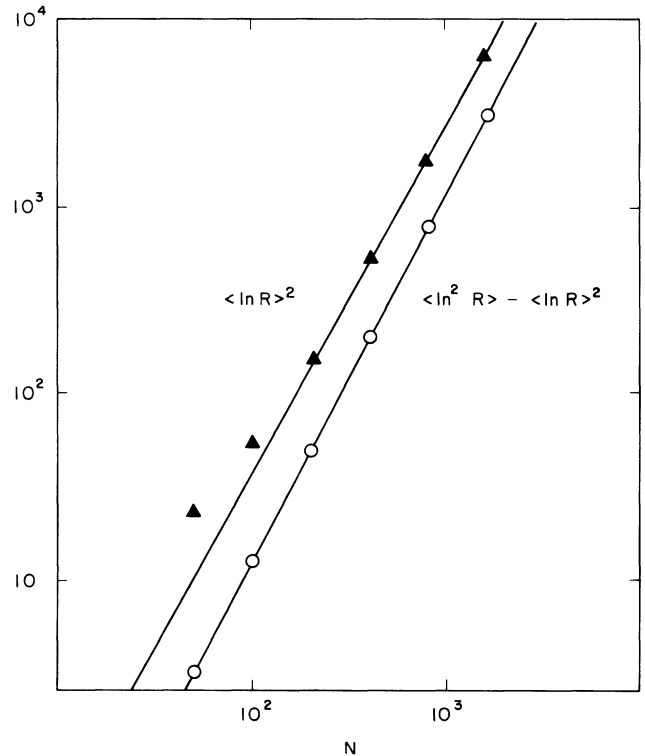


FIG. 2. Plot of $\langle \ln R_{\text{tot}} \rangle^2$ and $\langle \ln^2 R_{\text{tot}} \rangle - \langle \ln R_{\text{tot}} \rangle^2$ as functions of the size of the system N for $\epsilon = 0.5$ and $\beta = 1.5$ corresponding to $\lambda = 1$ (ferro case). We note that the best slope to $\langle \ln R_{\text{tot}} \rangle^2$ is slightly lower than the correct value due to the correction term discussed in the text.

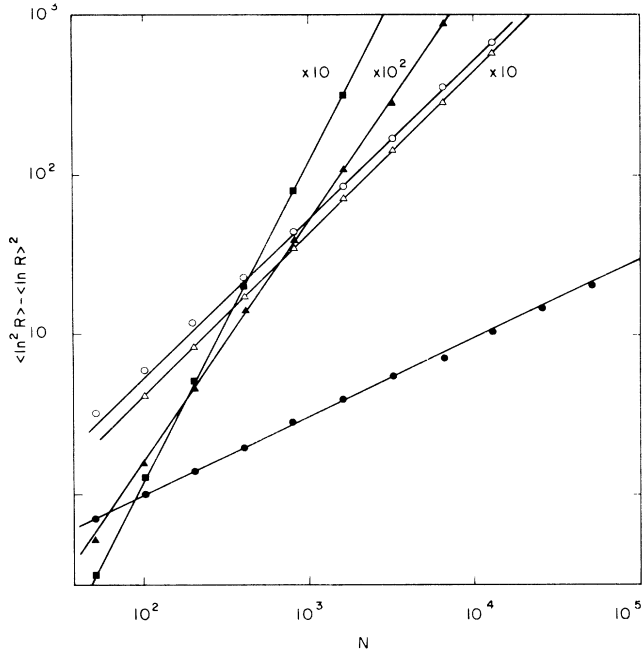


FIG. 3. Double logarithmic plot of $\langle \ln^2 R_{\text{tot}} \rangle - \langle \ln R_{\text{tot}} \rangle^2$ for $\epsilon = 0.5$ and various values of β . Ferro case: $\beta = 1.5$ (■), 2.5 (▲), and 3.5 (△). Antiferro case: $\beta = 1.5$ (●) and 2.5 (○).

fluctuation $\langle \ln^2 R_{\text{tot}} \rangle - \langle \ln R_{\text{tot}} \rangle^2$ for the ferro case with $\beta = 1.5$ (corresponding to $\lambda = 1$) as a function of N . Both curves have the same *asymptotic* slope, $1 + \lambda = 2$, as can be seen by following the procedure used to derive Eq. (10). For $\langle \ln R_{\text{tot}} \rangle^2$, the convergence to the predicted slope is slow due to the $\ln N$ correction in (10), while the fluctuations show the predicted slope already at small values of N .

Figure 3 shows the fluctuations of $\langle \ln R_{\text{tot}} \rangle$ for $\beta = 1.5$, 2.5, and 3.5 in the ferro case (corresponding to $\lambda = 1, \frac{1}{2}$, and 0) and for $\beta = 1.5$ and 2.5 in the antiferro case (corresponding to $\lambda = -\frac{1}{2}$ and 0). The numerical results are in excellent agreement with the predictions, Eqs. (10), (12), and (13).

Study of the above correlated resistor chain also provides insight into another interesting physical problem: diffusion in the presence of quenched *correlated* disorder. We shall see that this significantly generalizes the classic Sinai model for diffusion in the presence of uncorrelated random fields.⁸

Consider a random walker on a topologically one-dimensional system. The probabilities $W_{j,j \pm 1}$ of hopping from site j to its two neighbors are proportional to the inverse of the corresponding resistances between the sites (see Fig. 4).⁹⁻¹¹ Hence

$$W_{j,j-1}/W_{j,j+1} = (1 + \epsilon)^{\tau_j}, \quad (14)$$

where now ϵ plays the role of a *local bias field*. From the normalization condition $W_{j,j+1} + W_{j,j-1} = 1$, we ob-

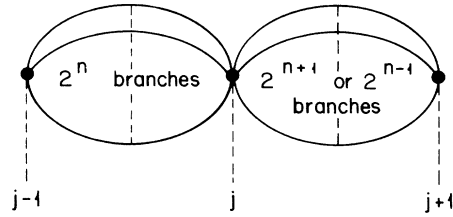


FIG. 4. The connection of the present model to diffusion in the presence of a spatially correlated bias field can be seen by connecting in series 2^{n_j} parallel equivalent branches between each node. The ratio of probabilities $W_{j,j+1}/W_{j,j-1}$ is the ratio of the corresponding number of branches on either side of node j .

tain $W_{j,j \pm 1} = (1 \pm E)/2$ where $E \equiv \epsilon/(2 + \epsilon)$.

The mean logarithm of the time the random walker takes to travel a distance L along the chain is proportional⁹⁻¹¹ to the fluctuations of the field biased against the walker,

$$\langle \ln t \rangle \sim X(L). \quad (15)$$

Accordingly, the first passage time in the diffusion problem plays a similar role as the resistance in the resistor problem. Correspondingly, when we take into account correlations between the local bias fields [which are determined by the correlations between the τ_i ; see Eq. (6)] we obtain on substituting (9) into (15)

$$\langle \ln t \rangle \sim L^{(1+\lambda)/2}. \quad (16)$$

The Sinai result is obtained for the average of the displacement for fixed t . If we assume that our result (16) will hold for fixing t and averaging L , we recover the Sinai result in the particular case $\lambda = 0$ (uncorrelated fields). The conditions under which the averages can be changed from fixed L to fixed t will be discussed elsewhere.

We thank S. Redner for helpful discussions. This work was supported in part by NATO, Minerva Gesellschaft für die Forschung m.b.H., NSF, ONR, and computer time from the Boston University Academic Computing Center. This work is based in part on the Ph.D. thesis of R. Blumberg Selinger (Harvard University).

¹There is much work on the effects of correlations in spatial disorder on the *nonclassical* properties of disordered but not completely random systems. One example is the Ziman theory of nonclassical transport in liquid and amorphous metals. See, e.g., the theoretical work of D. Nicholson and L. Schwartz, Phys. Rev. Lett. **49**, 1050 (1982), and references therein. Here we treat a rather different problem, that of providing an underlying microscopic model for correlated spatial disorder that can be solved exactly.

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⁵Although the main immediate interest in the model is *theoretical* (it is the first soluble model incorporating correlated spatial disorder), we note that there exist *experiments* where correlated spatial disorder is relevant, e.g., (i) measurements of the electrical resistance of elongated cylindrical rock samples saturated with salt water, as a function of length. One property of such systems that is already known experimentally

(Ref. 3) and is consistent with our model is the zero percolation threshold of such systems. (ii) Measurements of resistance to flow in clogged pipes (or "blood vessels"). The process of clogging involves deposition of material on the walls of the pipes, which is faster in and near regions where material is already deposited. The deposition profile may be expected to produce local resistances to flow that are more complex than simple random processes.

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