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Conformal Spectra of Polymers on a Random Surface

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We consider polymers (self-avoiding walks) on a randomly triangulated planar surface. The partition function of L polymer lines tied by their extremities, in a fluctuating metric, is calculated exactly. The exact infinite conformal spectra so derived are in complete agreement with the exponents found recently for 2D quantum gravity by Knizhnik, Polyakov, and Zamolodchikov.

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In the past few years, two distinct fields of statistical mechanics have been the object of considerable theoretical developments: conformal invariance for two-dimensional (2D) critical phenomena,[1,2] and the theory of random surfaces,[3-5] both being actually related to string theories.[3] Conformal theories are based on an algebra of operators,[1,2] whose scaling dimensions form the so-called Kac table,[6] and are in direct relation to the critical exponents of the associated statistical models. This has led to a blowing of exact results, and even to the hope of classifying all critical conformal theories in 2D. On the other hand, random surfaces have long been thought to be related to some 2D conformal theories,[3] but a direct relation was lacking. A simple discretization of the Polyakov string model was given by a model of dynamically triangulated surfaces, where an abstract 2D random simplicial lattice is embedded in d-dimensional space.[4,5] This equivalently describes the quantum gravity where the fluctuations of the metric are coupled to a matter d-component free field. Note that pure gravity is described by an ensemble of abstract (planar) graphs without embedding. In the study of critical properties of a random surface embedded in d space,[4,5] exact results have been obtained for the string susceptibility exponent γ_{string} in $d=-2$,[4] or $d=0$,[4,5]

$$\gamma_{\text{string}}=-1, \quad d=-2, \qquad \gamma_{\text{string}}=-\tfrac{1}{2}, \quad d=0. \tag{1}$$

Recently, new statistical models have been proposed by Kazakov, who solved the Ising model on a *random* 2D lattice,[7] and the $Q=0,1$ Potts models.[8] Thus a completely new field is now opened by the transfer of all standard 2D statistical models from the plane to a random surface, and the study of their critical properties. This corresponds to study phase transitions coupled to the fluctuations of the intrinsic metric of the 2D space, i.e., statistical mechanics in the presence of quantum gravity. Very recently, in two breakthrough papers,[9,10] Knizhnik, Polyakov, and Zamolodchikov (KPZ) succeeded in building the associated conformal theory describing the critical behavior of such systems. The usual conformal dimensions $\Delta^{(0)}$ of the Kac spectrum[6] in the plane are converted by the "gravitational dressing" into new ones Δ satisfying[9,10]

$$\Delta-\Delta^{(0)}=\Delta(1-\Delta)/g', \tag{2}$$

where g' is related to the central charge of the statistical model (the matter field) by[10]

$$c(\equiv d)=13-6g'-6/g'=1-6(1-g')^2/g' \tag{3}$$

($g'\equiv k+2=1-\gamma_{\text{string}}$ in KPZ[10]). For the massless Gaussian field describing the fluctuations of a surface embedded in d dimensions, $c=d$ is the number of degrees of freedom.

Now we remark precisely that the embedding dimensions $d=-2,0$ of Eq. (1) where an *exact solution* to random surfaces could be found[4,5] are also the central charges of dense ($c=-2$) or dilute polymers ($c=0$), e.g., self-avoiding walks (SAW) corresponding, respectively, to the $m=1$ and $m=2$ nonunitary minimal models just preceding the Friedan, Qiu, and Shenker classification for $m\geq 3$.[1] This suggests that an *exact* solution to *polymers* on a *random surface* may also exist. We

give indeed that solution here.

Before proceeding to the calculations, let us recall the conformal spectrum of polymers in the plane.[11] The key idea is to consider the correlation function G_L of L polymer lines of self-avoiding walks of fluctuating lengths tied together at their extremities $\mathbf{X}$ and $\mathbf{Y}$, i.e., a "*watermelon* network"[11,12] (Fig. 1). The total length of the L walks, controlled by a fugacity K, diverges at a critical value $K_c^{(0)}$, where the correlation function G_L decays algebraically like

$$G_L(\mathbf{X}-\mathbf{Y})=\langle\phi_L(\mathbf{X})\phi_L(\mathbf{Y})\rangle\sim|\mathbf{X}-\mathbf{Y}|^{-2x_L}, \tag{4}$$

where x_L is the scaling dimension of the conformal operator ϕ_L associated with the L-line polymer vertex.[11] For fugacities $K>K_c^{(0)}$, the SAW *fill* the plane (*dense* polymers), and new anomalous dimensions x_L^p appear.[11] In terms of the Kac table[1] $\Delta_{p,q}^{(0)}=[(m+1)p-mq]^2-1/4m(m+1)$ of central charge $c=1-6/m(m+1)$, one has[11]

$$\begin{aligned} &x_L=2\Delta_{L/2,0}^{(0)}=(9L^2-4)/48, \quad m=2, \quad c=0,\\ &x_L^p=2\Delta_{0,L/2}^{(0)}=(L^2-4)/16, \quad m=1, \quad c=-2. \end{aligned} \tag{5}$$

This can be generalized to the whole $O(n)$ model,[11] whose $n\to 0$ limit describes SAW.

Let us now consider polymers on a surface with fluctuating metric. We show in this Letter that this model is *exactly solvable*. We find the exact universal conformal dimensions dressed by gravity for the dilute and dense cases, respectively,

$$\Delta_L=(3L-2)/8, \quad \Delta_L^p=(L-2)/4. \tag{6}$$

It is easy to check that these results are among the solutions of KPZ Eqs. (2) and (3) where we insert the exact polymer results (5) in the plane for[11] $g'=1+1/m=\frac{3}{2}$ ($c=0$, $K=K_c^{(0)}$, dilute SAW) and $g'=2$ ($c=-2$, $K>K_c^{(0)}$, dense SAW). Let us now proceed to the exact solution.

Take as the ensemble of planar random lattices the set of all φ^3 graphs G with the topology of the sphere. Its partition function is

$$Z(\beta)=\sum_G \frac{1}{S(G)} e^{-\beta|G|}, \tag{7}$$

where $|G|$ is the number of vertices of the graph G, and $S(G)$ is the order of the symmetry group of G. The series (7) converges for all values of the "cosmological constant" β larger than some critical β_c. At $\beta\to\beta_c^+$, a singularity due to *infinite* graphs appears and the singular part of the string susceptibility behaves as

$$\chi=\partial^2 Z/\partial\beta^2\sim(\beta-\beta_c)^{-\gamma_{\text{string}}}, \tag{8}$$

with $\gamma_{\text{string}}=-\frac{1}{2}$ [Eq. (1) for $d=0$].

Further, we shall need the partition function of random graphs with n external legs

$$G_n(\beta)=\sum_{n\text{ leg planar }G} e^{-\beta|G|}. \tag{9}$$

This quantity was calculated in the beautiful paper[13] through the large-N limit of an $N\times N$ matrix integral. It has an integral representation

$$G_n(\beta)=\int_{2a}^{2b} d\lambda\rho(\lambda)\lambda^{(n)}, \tag{10}$$

where $\rho(\lambda)$ is the density of eigenvalues[13]

$$\begin{aligned} \rho(\lambda)=(1/2\pi)[1-e^{-\beta}(a+b+\lambda)]\\ \times[(\lambda-2a)(2b-\lambda)]^{1/2}, \end{aligned} \tag{11}$$

and a and b are functions of β through

$$\begin{aligned} &a+b=\sigma e^{\beta}, \quad 2e^{-2\beta}=\sigma(1-\sigma)(1-2\sigma),\\ &\delta\equiv a-b=2(1-2\sigma)^{-1/2}. \end{aligned} \tag{12}$$

The value of the critical fugacity $e^{-\beta_c}=(12\sqrt{3})^{-1/2}$ is found from $d\beta/d\sigma=0$.

Let us now put self-avoiding walks on the abstract random graphs. We consider the watermelon polymer network[11,12] with L polymer lines $\Gamma_{ij}^{(l)}$ ($l=1,\ldots,L$) going from the point i to the point j of the random lattice along the links of the lattice (Fig. 2). The sum is performed over all polymer configurations and planar φ^3 lattices

$$Z_L(\beta,K)=\sum_G \frac{e^{-\beta|G|}}{S(G)} \sum_{i,j\in G} \sum_{\substack{\Gamma_{ij}^{(l)}\subset G\\ l=1,\ldots,L}} K^{|\Gamma|}, \tag{13}$$

where K is the fugacity associated with the total length $|\Gamma|$ of the L SAW's. To calculate this quantity, we apply the same trick as for calculating maximal trees on a random 2D lattice.[4] We *first* sum over all the planar graphs which form the L connected pieces of the lattice bounded by two successive polymer lines $\Gamma^{(l)}$ and $\Gamma^{(l+1)}$

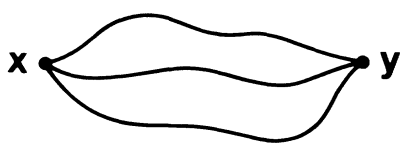

FIG. 1. L-watermelon network in the plane ($L=3$).

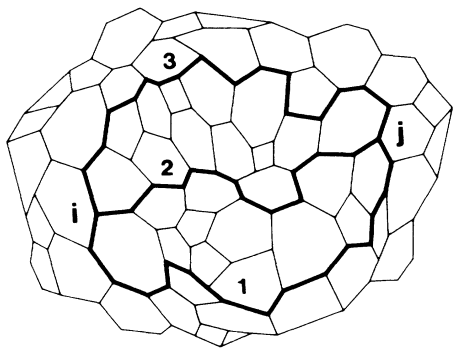

FIG. 2. $L=3$ polymer lines on a randomly triangulated spherical lattice, made of φ^3 vertices.

(recall that G has the topology of the sphere and $\Gamma^{(L)}$ and $\Gamma^{(1)}$ are also the boundaries of the external domain on Fig. 2), and *then* over the lengths of the polymers. We find

$$Z_L(\beta,K)=\frac{K^L}{L}\sum_{\{m_l,n_l\}=0}^{\infty}(e^{-\beta}K)^{m_1+n_1+\cdots+m_L+n_L}\binom{m_1+n_1}{m_1}G_{n_1+m_2}(\beta)\times\binom{m_2+n_2}{m_2}G_{n_2+m_3}(\beta)\cdots\binom{m_L+n_L}{m_L}G_{n_L+m_1}(\beta), \quad (14)$$

where the (m_l,n_l), $l=1,\ldots,L$, are the numbers of legs or lines attached on the lth SAW and belonging to the adjacent random graphs on both sides. The random graph located between SAW lines l and $l+1$ contributes a n-point function G_n to Eq. (9) with $n=n_l+m_{l+1}$ (Fig. 3). This convolutive sum (14) is now easily performed by using the integral representation (10)

$$Z_L(\beta,K)=\frac{K^L}{L}\int_{2a}^{2b}\cdots\int_{2a}^{2b}\prod_{l=1}^{L}\rho(\lambda_l)d\lambda_l\frac{1}{1-z(\lambda_1+\lambda_2)}\frac{1}{1-z(\lambda_2+\lambda_3)}\cdots\frac{1}{1-z(\lambda_L+\lambda_1)}, \quad (15)$$

where $z\equiv e^{-\beta}K$. Each of the denominators is now represented by a Schwinger-type integral with α parameters, and the integrals over the λ's factorize immediately

$$Z_L(\beta,K)=\frac{K^L}{L}\int_0^{\infty}\prod_{l=1}^{L}d\alpha_l\exp\left(-\sum_{l=1}^{L}\alpha_L\right)\prod_{l=1}^{L}\mathcal{F}[z(\alpha_l+\alpha_{l+1})], \quad (16)$$

where

$$\mathcal{F}(x)\equiv\int_{2a}^{2b}d\lambda\,\rho(\lambda)e^{x\lambda}, \quad (17)$$

and the convention $\alpha_{L+1}\equiv\alpha_1$ is used. The form (11) of $\rho(\lambda)$ allows an explicit calculation of the Laplace transform (17):

$$\mathcal{F}(x)=e^{x\sigma e^{\beta}}F(x\delta),$$
$$F(\mu)=\frac{\delta^2}{2}\left(1-2\sigma-\delta e^{-\beta}\frac{d}{d\mu}\right)\frac{I_1(\mu)}{\mu}, \quad (18)$$

where I_1 is the modified Bessel function of order 1.

We are interested in the singularities of $Z_L(\beta,K)$ (16) when the polymer and surface fugacities K and $e^{-\beta}$ approach their critical values K_c and $e^{-\beta_c}$, respectively.

The *polymer* critical point K_c for any fixed β corresponds in integral (16) to all α_l parameters becoming *large* simultaneously. Indeed, the asymptotic behavior of where the polymers and the surface become critical (infinite).
$\mathcal{F}(x)$ (18) reads explicitly

$$\lim_{x\to\infty}\mathcal{F}(x)=\frac{1}{2}\left(\frac{\delta}{2\pi}\right)^{1/2}e^{x(\delta+\sigma e^{\beta})}\left[(1-2\sigma-\delta e^{-\beta})x^{-3/2}+\left(\frac{3\sigma}{4\delta}+\frac{15}{8}e^{-\beta}\right)x^{-5/2}+O(x^{-7/2})\right]. \quad (19)$$

Thus integral (16) diverges when the entropic exponential growth of (19) is balanced, i.e., for

$$z_c\equiv e^{-\beta}K_c(\beta)=\tfrac{1}{2}(\delta+\sigma e^{\beta})^{-1}. \quad (20)$$

Note that z_c, i.e., K_c depends on β only [see (12)] and that $K_c(\beta)>K_c(\beta_c)$ for $\beta>\beta_c$. If we thus keep the polymer fugacity K below its *lowest* critical threshold $K_c(\beta_c)$, and let $\beta\to\beta_c^+$, we recover the usual singularity of the partition function (16) due to $\rho(\lambda)$ (11), i.e., that of the infinite planar lattice alone. In this case the polymers remain finite and do not change the critical behavior of the random graph, as given in (8).

Now, let us consider the inverse case, e.g., fix the "cosmological constant" $\beta>\beta_c$ and let the polymer fugacity $K\to K_c^-(\beta)$. The polymers now form a *dense* critical phase filling the random lattice, whose area is thus conjugate to K instead of β. The volume of the lattice is essentially that of the polymers. When $z\to z_c^-$ (20), the singular behavior of $Z_L(\beta,K)$ (16) is obtained by usual power counting on the α's in the integral representation (16)

$$Z_L(\beta,K)\sim(K_c-K)^{L/2};\quad K\to K_c^-(\beta),\quad \beta>\beta_c, \quad (21)$$

where only the first term of the asymptotic expansion (19) contributes to dominant order in this dense phase.

This does not hold true any more when its amplitude vanishes: $1-2\sigma-\delta e^{-\beta}=0$. This can be seen from (12) to be exactly the condition $d\beta/d\delta=0$ determining the critical point of the random lattices in the absence of polymers; hence $e^{-\beta_c}=(12\sqrt{3})^{-1/2}$. When $\beta\to\beta_c^+$, the coefficient $1-2\sigma-\delta e^{-\beta}$ vanishes as $(\beta-\beta_c)^{1/2}$. Near that point we get a *new* critical behavior of $Z_L(\beta,K)$, which is a consequence of the confluence of the singularities due to infinite polymers ($K\to K_c$) and infinite planar graphs ($\beta\to\beta_c$). Keeping both terms in (19) gives

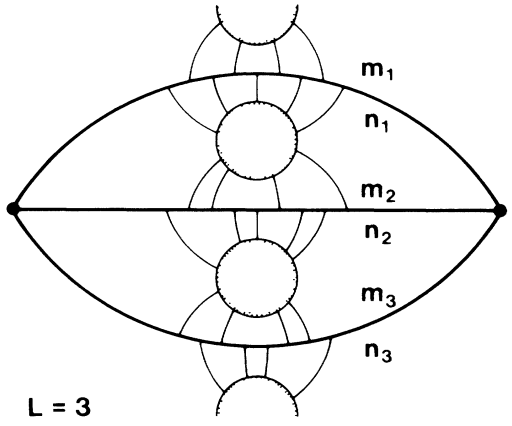

FIG. 3. Graphical representations of the convolutive sum (14), each bubble standing for a n-point correlator G_n.

the singular behavior of the integral (16) for large α's

$$Z_L(\beta,K)\sim\{(K_c-K)^{1/2}(\beta-\beta_c)^{1/2} + \text{const}(K_c-K)^{3/2}\}^L. \quad (22)$$

The double critical limit is obtained for $\beta\to\beta_c\sim(K_c-K)^2$ where the two terms are equivalent; hence

$$Z_L\sim(K_c-K)^{3L/2}\sim(\beta-\beta_c)^{3L/4}. \quad (23)$$

Both polymers and nonfilled surface become infinite simultaneously (dilute polymers), and their respective mean sizes $|G|\sim(\beta-\beta_c)^{-1}$, $|\Gamma|\sim(K_c-K)^{-1}$, are related by

$$\beta-\beta_c\sim(K_c-K)^{\nu D},\quad \nu D=2, \quad (24)$$

where ν is the polymer size exponent, and D is the fractal dimension of the surface. From (22) we find $\nu D=2$. Hence the mass of the polymer grows as a power of that of the random lattice $|\Gamma|\sim|G|^{1/\nu D}=|G|^{1/2}$, and the occupied fraction $|\Gamma|/|G|$ vanishes at the critical point. This is the phase of *dilute* polymers. Note that for dense polymers [Eq. (21)], the polymer and lattice critical volumes are the same and $\nu D=1$. Note also that (24) corresponds to the finite-size scaling regime near the dilute critical point.

Let us now relate these sets of exact critical exponents (21), (23), and (24) to the gravitational conformal dimensions Δ_L (6). We have to *normalize* the partition function Z_L (13) by that of the random surface with two rooted points, i.e., by $Z''(\beta)$ (8), in order to interpret it as a correlation function of conformal fields as in (4): $G_L=Z_L(\beta,K)/Z''(\beta)$. We expect it to scale like $G_L\sim(\beta-\beta_c)^{2\Delta_L}$ in the finite-size scaling regime where $\beta-\beta_c\sim(K_c-K)^{\nu D}$, with $\nu D=(1-\Delta_2)^{-1}$, Δ_2 being here the energy anomalous dimension. From these definitions, and using the string susceptibility (8), we find

$$Z_L\sim|K_c-K|^{\nu D(2\Delta_L-\gamma_{\text{string}})}. \quad (25)$$

For dense polymers [Eq. (21)], we find $\nu D=1$, $\gamma_{\text{string}}=-1$, and $\Delta_L=(L-2)/4$, while for dilute polymers [Eq. (23)], $\nu D=2$, $\gamma_{\text{string}}=-\frac{1}{2}$, and $\Delta_L=(3L-2)/8$, as announced in (6), QED. Note that these exact results lift the indeterminacy of KPZ formulas (2) and (3). Note also that the result (21) for a dense ringlike polymer ($L=2$) gives a new derivation of $\gamma_{\text{string}}=-1$ for $c=d=-2$, which was obtained in Ref. 4 from spanning trees. This shows that on a random lattice dense polymers and spanning trees are in the same universality class[14] as noticed for the plane.[11]

This study can be generalized to the $O(n)$ loop model[15] parametrized in Coulomb gas formalism by $n=-2\cos\pi g$, where $g\in[0,1]$ in the low-temperature phase in the plane $K>K_c^{(0)}$ and $g\in[1,2]$ at $K_c^{(0)}$.[15] Then the dimensions $\Delta_L^{(0)}$ in the plane are[11,15] for watermelon correlation functions $\Delta_L^{(0)}=gL^2/16-(1-g)^2/4g$ and in the presence of gravity [Eqs. (2) and (3)]

$$\Delta_L^D=\frac{L}{4}-\frac{1}{2}\left[\frac{1}{g}-1\right],\quad \beta>\beta_c,\quad K\to K_c,\quad g\in[0,1],$$
$$\Delta_L=g\frac{L}{4}-\frac{1}{2}(g-1),\quad \beta\to\beta_c,\quad K\to K_c,\quad g\in[1,2]. \quad (26)$$

Note that g' in (2) and (3) reads $g'=1/g$ in the dense phase, $g'=g$ at the confluent dilute critical point, and in both cases $g'=1+1/m$. The polymer values (6) are recovered for the $O(n=0)$ model, $g=\frac{1}{2}$ for dense SAW, $g=\frac{3}{2}$ for dilute ones. Direct calculations of these $O(n)$ dimensions on the random lattice are in progress,[16] as well as for polymers of any topology.[14]

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[1]A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984); D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).

[2]J. L. Cardy, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1987), Vol. 11.

[3]See, e.g., A. M. Polyakov, *Gauge Fields and Strings* (Harwood-Academic, Chur, 1987).

[4]D. V. Boulatov, V. A. Kazakov, I. K. Kostov, and A. A. Migdal, Nucl. Phys. **B275 [FS17]**, 641 (1986).

[5]F. David, Nucl. Phys. **B257 [FS14]**, 45,543 (1985); J. Ambjorn, B. Durhuus, and J. Fröhlich, Nucl. Phys. **B257**, 433 (1985).

[6]V. G. Kac, in *Group Theoretical Methods in Physics,* edited by W. Beiglböck, A. Böhm, and E. Takasugi, Lecture Notes in Physics Vol. 94 (Springer-Verlag, Berlin, 1979), p. 441.

[7]V. A. Kazakov, Phys. Lett. A **119**, 140 (1986); D. V. Boulatov and V. A. Kazakov, Phys. Lett. B **186**, 379 (1987).

[8]V. A. Kazakov, Nucl. Phys. B (Proc. Suppl.) **4**, 93 (1988), and to be published.

[9]A. M. Polyakov, Mod. Phys. Lett. A **2**, 893 (1987).

[10]V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov,

to be published.

[11]B. Duplantier and H. Saleur, Nucl. Phys. **B290 [FS20]**, 291 (1987); H. Saleur, J. Phys. A **20**, 455 (1987), and **19**, L807 (1986); B. Duplantier, J. Stat. Phys. **49**, 411 (1987).

[12]B. Duplantier, Phys. Rev. Lett. **57**, 941 (1986).

[13]E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978).

[14]B. Duplantier and I. Kostov, to be published.

[15]B. Nienhuis, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1987), Vol. 11.

[16]I. Kostov, to be published.

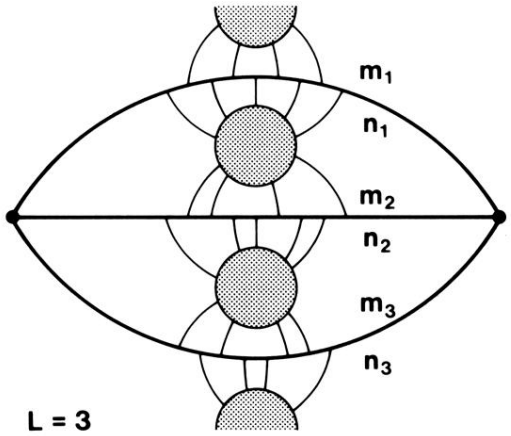

FIG. 3. Graphical representations of the convolutive sum (14), each bubble standing for a n-point correlator G_n.