Particlelike Solutions of the Einstein-Yang-Mills Equations

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We study the static spherically symmetric Einstein-Yang-Mills equations with SU(2) gauge group and find numerical solutions which are nonsingular and asymptotically flat. These solutions have a high-density interior region with sharp boundary, a near-field region where the metric is approximately Reissner-Nørdstrom with Dirac monopole curvature source, and a far-field region where the metric is approximately Schwarzschild.

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Introduction.— Neither the vacuum Einstein equations nor the pure Yang-Mills equations have nontrivial static globally regular (i.e., nonsingular, asymptotically flat) solutions. For pure Yang-Mills fields, this was shown by Deser and Coleman,[1,2] and Deser has also shown that there are no static Einstein-Yang-Mills (EYM) solutions in $D=2+1$.[3] The corresponding result for vacuum gravity and for Einstein-Maxwell fields is Lichnerowicz's theorem, with modifications permitting interior horizons due to Israel,[4] Robinson,[5] Bunting and Masood-ul-Alam.[6] So it is natural to conjecture that the coupled EYM equations also have no nontrivial globally regular solutions.

In this paper we present strong (numerical) evidence for the existence of a discrete family of globally regular solutions of the static EYM equations— the gravitational attraction can balance the Yang-Mills (YM) repulsive force. The solutions we find have gauge group SU(2) and are spherically symmetric. Their behavior shows three distinct regions, with two transition zones. The interior region, $r<1$, is characterized by high YM curvature and stress-energy. The transition zone about $r=1$ is marked by large fluctuations in the connection and metric coefficients. These fluctuations are rapidly damped in the near-field region, $1<r<R_0$, where the YM curvature decays polynomially and has dominant behavior modeled closely on the Dirac magnetic monopole and the metric is very close to the extremal Reissner-Nørdstrom solution.[7] In the "charge-shielding" transition zone, $R_0<r<R_1$, the Reissner-Nørdstrom charge gradually decays to zero, and in the far-field region, $r>R_1$, the solution approximates the Schwarzschild solution, with zero YM charge integrals.[8]

There are two important lessons to be drawn from these results:

(1) The gravitational interaction cannot be dismissed as too weak to be of consequence— the existence of these solutions depends essentially on the interaction between the YM and Einstein equations.

(2) Both nontrivial Yang-Mills vacua and "symmetry breaking" can occur without involvement of the Higgs mechanism. The equations contain no Higgs fields and are topologically trivial, yet these solutions have nonzero YM curvature, which progressively degenerates su(2) $\to$ u(1) $\to 0$ as $r\to\infty$.

EYM equations and boundary conditions.— The spherically symmetry SU(2) connection has been described in many places.[9-12] If we let τ_i, $i=1,2,3$, denote the usual basis of su(2), θ and ϕ the usual polar coordinates on S^2, and parametrize the space of S^2 orbits of the symmetry group by (r,t), the connection can be written

$$A=a\tau_3 dt+b\tau_3 dr+(c\tau_1+d\tau_2)d\theta+(\cot\theta\tau_3+c\tau_2-d\tau_1)\sin\theta d\phi, \quad (1)$$

where a, b, c, and d are functions of (r,t). This connection arises from the global symmetry group SU(2) rather than SO(3).[13]

By changing coordinates, we can write the spherical metric as

$$ds^2=-T^{-2}dt^2+R^2dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \quad (2)$$

We consider only time-independent solutions in this paper, and so the functions a, b, c, d, R, and T depend now only on r.

The connection A has a residual U(1) gauge freedom, $A\to h^{-1}Ah+h^{-1}dh$ where $h(r)=\exp(\psi\tau_3)$, which we use to impose the radial gauge $b\equiv 0$. The YM curvature is $F=dA+A\wedge A$ and the YM equations $d*F=0$ are

$$(r^2R^{-1}Ta')'-2RT(c^2+d^2)a=0,$$

$$(R^{-1}T^{-1}c')'+RTa^2c+r^{-2}RT^{-1}(1-c^2-d^2)c=0,$$

$$cd'-dc'=0.$$

Using the remaining gauge freedom we can set $d=0$, $c=w\in\mathbb{R}$ with $w\geq 0$ at $r=0$. The EYM equations derived from the Lagrangean $\int(-R+|F|^2)\sqrt{g}dx$, $|F|^2=g^{ab}g^{cd}F_{ac}F_{bd}$, are

$$\mathrm{Ric}_{ab}=2F_{ac}F^c_b-\tfrac{1}{2}|F|^2g_{ab}.$$

We now assume $a \equiv 0$, so that the YM curvature is purely magnetic and there are not dyons.[14] In fact, one can show that this assumption follows from suitable asymptotics and finite energy. If we introduce $m(r)$ by $R=(1-2m/r)^{-1/2}$, the static spherically symmetric EYM equations reduce to

$$m' = (1-2m/r)w'^2 + \tfrac{1}{2}(1-w^2)^2/r^2, \tag{3}$$

$$r^2(1-2m/r)w'' + [2m-(1-w^2)^2/r]w' + (1-w^2)w = 0, \tag{4}$$

with the supplementary equation

$$2r(1-2m/r)T'/T = (1-w^2)^2/r^2 - 2(1-2m/r)w'^2 - 2m/r, \tag{5}$$

and YM curvature tensor

$$F = w'\tau_1 dr\wedge d\theta + w'\tau_2 dr\wedge \sin\theta d\phi - (1-w^2)\tau_3 d\theta\wedge\sin\theta d\phi. \tag{6}$$

We introduce the radial and angular magnetic curvatures,

$$B_L = -\frac{(1-w^2)}{r^2}\tau_3, \quad B_T = \frac{w'}{rR}\tau_1,$$

and note that the local energy density is

$$4\pi T_{00} = |B_T|^2 + \tfrac{1}{2}|B_L|^2 = m'/r^2. \tag{7}$$

We impose the asymptotic conditions $m(r)\to M<\infty$ as $r\to\infty$, so that $R(r)\to 1$, and $T(r)\to 1$. There are two explicit solutions satisfying these conditions. If $w\equiv 1$, then

$$m(r) = M, \text{ constant, and } R = T = (1-2m/r)^{-1/2}.$$

which is just the Schwarzschild metric, with vanishing YM curvature. If $w\equiv 0$, then $a(r)=a=a_0-e/r$, a_0, e constant, and

$$2m(r) = 2M-(1+e^2)/r, \quad R=T=(1-2m/r)^{-1/2}.$$

This describes the Reissner-Nørdstron (RN) metric with mass M, electric charge e, magnetic charge $g=1$, and YM curvature

$$F = (e/r^2)\tau_3 dr\wedge dt - \tau_3 d\theta\wedge\sin\theta d\phi.$$

Since F is u(1) valued, with $e=0$, this represents a Dirac monopole source for the RN metric, a solution also noted by Harnad, Shnider, and Tafel.[15]

Of course, the RN and Schwarzschild solutions both have singularities. We now consider the boundary conditions at $r=0$ needed for nonsingular solutions. The requirement that the density T_{00} be finite implies

$$2m(r) = O(r^3), \tag{8}$$

$$w(r) = 1+O(r^2), \quad w'(r) = O(r) \text{ as } r\to 0. \tag{9}$$

Together with $T'(0)=0$, this ensures that the metric is regular at $r=0$, and a result of Uhlenbeck now shows that the bundle extends smoothly across $r=0$.

There are some elementary observations about these equations. We have

$$(\ln R/T)' = 2w'^2/r,$$

and so R/T is increasing and $T>R\geq 1$. Since T satisfies

$$[r^2/R(1/T)']' = 2(1-2m/r)w'^2/RT + (R/r^2T)(w^2-1)w^2,$$

we see that $T'<0$ if $w^2 \not\equiv 1$. If we write the equation for w in the form

$$\tfrac{1}{2}[(w^2)'/RT]' = w'^2/RT + (R/r^2T)(w^2-1)w^2, \quad w^2\not\equiv 1,$$

either $w^2<1$ or $w^2\not\equiv 1$ if the total mass is finite. Also, $ww''<0$ if $w'=0$ and $w^2<1$, which is consistent with w oscillating.

Numerical solutions.—Our method of finding nontrivial solutions of (3)–(9) is quite simple minded. Using the formal power-series expansion about $r=0$,

$$2m = 4b^2r^3 + \tfrac{16}{5}b^3r^5 + O(r^7),$$

$$w = 1 + br^2 + (\tfrac{4}{5}b^3 + \tfrac{3}{10}b^2)r^4 + O(r^6), \quad b\in R,$$

we construct initial data at $r=0.01$ for a standard-package ordinary-differential-equation solver. By adjusting the free parameter b, we "shoot" global solutions. With tolerance 10^{-12}, the solutions were found to vary continuously and regularly with b, indicating that this numerical procedure is well behaved.

For $b<-0.0706$, the solver broke down at $r<1$ with w' large and negative. For $-0.706<b<0$, the generic solution oscillated in $|w|<1$ before crossing $|w|=1$ and rapidly going to infinity. However, at a discrete set of values in this range, the solutions after oscillating are seen to be asymptotic to $w=\pm 1$ (Fig. 1). We can index these solutions by $k=$number of zeroes of w. There are three distinguished regions: the inner core region I, $r>1$; the near-field region II, $r>1$, $w\sim 0$; and the far-field region III, $r\gg 1$, $w\sim\pm 1$.

The stress-energy density (7) is large in the inner core but decays rapidly (Fig. 2). In region II, $B_L \gg B_T$, indicating that the solution approximates the U(1) Dirac monopole.

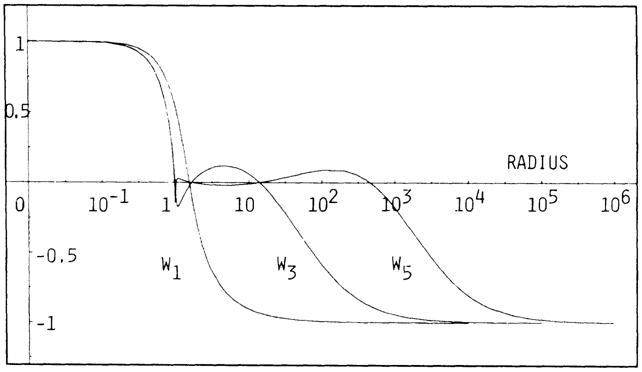

FIG. 1. Connection parameter w_k, $k=1,3,5$.

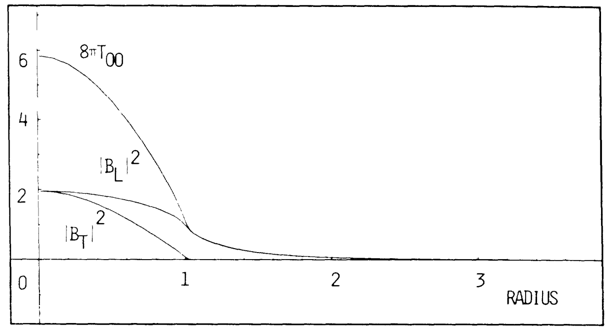

FIG. 2. Energy density in the interior, $k=3$.

The mass function $m(r)$ increases to a limit $M_k > 0$, where the total mass M_k was found empirically to satisfy $M_k = 1 - 1.055e^{-1.420k}$. The metric coefficient $R = (1 - 2m/r)^{-1/2}$ has a large peak at $r \sim 1$, the size of the peak depending on k. As suggested above, in the near-field region II, R approximates the RN coefficient with $e^2 = 0$ and magnetic charge $g^2 = 1$. This can be clearly seen by definition of the RN (magnetic) charge,

$$g^2(r) = 2r(M_k - m).$$

Notice (Fig. 3) that the charge is shielded in the transition between regions II and III, and in the far-field region III the metric is approximately Schwarzchild with mass M_k. EYM-Higgs perturbations of the extremal RN metric about the internal infinity have been studied by Hajicek[16] and may apply to model the behavior near $r=1$.

In the far-field region we have the asymptotic expansion, for some $c > 0$

$$w \sim 1 - c/r, \quad m \sim M - c^2/r^3,$$

$$|B_T| \sim 2c/r^3, \quad |B_L| \sim c/r^3.$$

Thus the YM curvature decays polynomially and the total YM charge integrals[8] vanish.

Discussion.— Although the model has an unspecified length scale (we have $c = G = 1$), the ratio mass radius of these solutions is fixed $\simeq 1$ (by radius we mean the radius of the inner-core region). Assignment of the particle radius 1 Planck length (1.6×10^{-33} cm) leads to a mass of 1 Planck mass (2.2×10^{-5} g). Changing of the coupling constant does not affect this ratio: The Lagrangean $\int(-R + \lambda^2 |F|^2)\sqrt{g}$ has the well-known scaled solutions[2]

$$m_\lambda(r) = \lambda m(r/\lambda), \quad w_\lambda(r) = w(r/\lambda), \quad T_\lambda(r) = T(r/\lambda),$$

where m, w, and T are static spherical solutions for the original Lagrangean. This rescaling does, of course, affect the YM curvature— the rescaled magnetic field satisfies

$$B_\lambda(r) = \lambda^{-2} B(r/\lambda).$$

There are strong similarities with magnetic monopole models,[17] but there are also some significant differences. In the near-field region, the dominant contribution B_L to the YM curvature closely approximates the U(1) Dirac monopole, but the transverse component B_T, although small, appears to decay polynomially and not exponentially. In the far-field region the curvature decay is still polynomial but $O(r^{-3})$ rather than the U(1) monopole $O(r^{-2})$, and so the YM charges vanish trivially.

If we take stability to mean that the Hessian of the energy functional,

$$I(w) = \int (|B_L|^2 + 2|B_T|^2) r^2 dr, \qquad (10)$$

is positive definite, then simple heuristics based on the observed numerical behavior (Fig. 1) indicate that at best only the $k=1$ and $k=2$ solutions can be stable. If we assume $w \sim 0$ on the interval [1;8], the Hessian can be approximated by

$$\delta^2 I(w)(\phi,\phi) \approx \int_0^\infty \left[\phi'^2 - \frac{1}{r^2}\phi^2\right] dr$$

for $\phi \in C_0^\infty([1;8])$. We readily see that $\delta^2 I$ has negative eigenvalues by testing with the continuous, piecewise linear function $\phi = 1$ for $2 < r < 4$, $=0$ for $r < 1, r > 8$, and linear elsewhere. This indicates that the solutions with $k \geq 3$ are unstable, as the near-field region where $w \sim 0$ then includes [1;8].

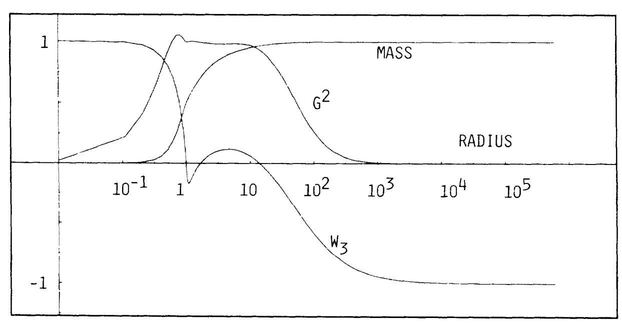

FIG. 3. Mass, RN charge, and w for $k=3$.

By analogy with the superposition solutions for extremal RN solutions, we may expect superposition solutions for the static EYM equations as well. Finally, we mention the review article of Malec,[18] which presents some "no go" results for the static EYM equations. These results do not, of course, cover our solution, but do show that it is a strictly nonperturbative effect.

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[1]S. Deser, Phys. Lett. **64B**, 463–465 (1976).

[2]S. Coleman, in *New Phenomenon in Subnuclear Physics,* edited by A. Zichichi (Plenum, New York, 1975).

[3]S. Deser, Class. Quantum. Grav. **1**, L1–L2 (1984).

[4]W. Israel, Commun. Math. Phys. **8**, 245–260 (1968).

[5]D. Robinson, Gen. Relativity Gravitation **8**, 695–698 (1977).

[6]G. Bunting and A. Masood-ul-Alam, Gen. Relativity Gravitation **19**, 147–154 (1987).

[7]S. Hawking and D. Ellis, *The Large-Scale Structure of Spacetime* (Cambridge Univ. Press, Cambridge, 1973).

[8]P. Chrusciel and W. Kondracki, Phys. Rev. D **36**, 1874–1881 (1987).

[9]E. Witten, Phys. Rev. Lett. **38**, 121–124 (1977).

[10]P. Forgacs and N. Manton, Commun. Math. Phys. **72**, 15–35 (1980).

[11]J. Harnad, S. Shnider, and L. Vinet, J. Math. Phys. (N.Y.) **21**, 2719–2724 (1980).

[12]J. McKinnon, Honors thesis, Australian National University, 1987 (unpublished).

[13]R. Bartnik, to be published.

[14]B. Julia and A. Zee, Phys. Rev. D **11**, 2227–2232 (1975).

[15]J. Harnad, S. Shnider, and J. Tafel, J. Math. Phys. (N.Y.) **21**, 2236–2240 (1980).

[16]P. Hajicek, J. Phys. A **16**, 1191–1205 (1983).

[17]A. Jaffe and C. Taubes, *Vortices and Monopoles* (Birkhäuser, Boston, 1980).

[18]E. Malec, Acta Phys. Pol. B **18**, 1017–1064 (1987).