Action Principle and Partition Function for the Gravitational Field in Black-Hole Topologies

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Statistical mechanics of gravitational fields describing the black-hole topological sector, and the correspondence to thermodynamics, are considered. The Euclidean action is evaluated on the constraint hypersurface and a measure is obtained, resulting in a path-integral form of the canonical partition function. We obtain the usual black-hole entropy plus quantum corrections when the temperature and size of the system are appropriate. Under other conditions, we give evidence for the existence of a phase transition (change of topology).

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Recently there has been considerable progress in showing that black-hole thermodynamics can be derived from the statistical mechanics of the relevant gravitational fields. Zurek and Thorne[1] gave a statistical argument consistent with the Bekenstein-Hawking formula[2] for the entropy $S_{\rm BH}$ of a black hole. Making explicit use of the boundary conditions of the canonical ensemble, one of us showed that thermodynamically stable black-hole solutions of the Einstein equations do exist.[3] This result led to a successful treatment of an idea proposed by Gibbons and Hawking[4] to use the classical Euclidean action of black-hole solutions to obtain a "zero-loop" approximation to the partition function.[3] Braden and the present authors[5] then obtained in the same approximation a well-behaved expression for the density of states of the gravitational field in black-hole topologies.

In the present paper we go beyond the previous works, which all relied explicitly on properties of the classical black-hole geometries. We consider all regular spherically symmetric spacetimes of suitable topology in a certain class specified in detail below. Almost all of these geometries fail to satisfy the Einstein equations by a wide margin. We are going to include their effects in constructing a partition function for the fixed "single-black-hole topological sector." The ensemble is specified by the area $4\pi r_B^2$ of the "box" with a black hole at the center and by the inverse temperature β at $r=r_B$. In the Euclidean description adopted here, the four-geometries have topology $R^2\times S^2$, boundaries $S^1\times S^2$, and Euler characteristic $\chi=2$. We construct a "reduced" Euclidean action I_* for these geometries by eliminating the constraints in the Euclidean action. The remarkable form assumed by I_* suggests a measure and enables us to formulate the partition function Z effectively as a functional integral. This particular Z can be evaluated as an ordinary integral and is finite even though the number of four-geometries being summed is roughly as large as the number of smooth single-valued functions on a disk in R^2. [We use units in which $k_{\rm B}=c=1$. The Planck radius is $r_{\rm P}=(G\hbar)^{1/2}$ and the Planck mass is $M_{\rm P}=(\hbar/G)^{1/2}$.]

We adopt for the black-hole geometries metrics of the form

$$ds^2=U(\tau,y)d\tau^2+V^{-1}(y)dy^2+r^2(y)d\Omega^2, \tag{1}$$

where the radial coordinate $y\in[0,1]$ and the Euclidean time τ has a period 2π. U is a periodic in τ and has a prescribed constant value U_B at the boundary $y=1$, as does $r(1)\equiv r_B$. The three-geometry of the boundary is thus fixed and related to the boundary conditions of the canonical ensemble by

$$\beta\hbar=\int_0^{2\pi}U_B^{1/2}d\tau=2\pi U_B^{1/2}, \tag{2}$$

where β is the inverse temperature at the boundary of prescribed area $A_B=4\pi r_B^2$. The "center" of the geometry at $r(0)\equiv r_+$ is required to be regular. Thus, for each y-τ plane to be smooth in the product manifold, we impose

$$[V^{1/2}(U^{1/2})']_{y=0}=1, \tag{3}$$

where a prime denotes differentiation with respect to y. In addition, to distinguish in (1) "hot flat space" ($S^1\times R^3$) from the black-hole sector, we examine the Euler characteristic of the four-geometry, which in the present case is given by

$$2[V^{1/2}(U^{1/2})'(1-\hat V)]_{y=0}=2[1-\hat V(0)]. \tag{4}$$

The equality in (4) follows from (3) and we have defined $\hat V=V(r')^2$. Black holes have the Euler characteristic $\chi=2$, from which (4) yields the requirement $\hat V(0)=0$ in the black-hole sector.

With the three-geometry of the boundary ∂M fixed, the appropriate action is

$$I=-\frac{1}{16\pi G}\int_M R g^{1/2}d^4x+\frac{1}{8\pi G}\int_{\partial M}(K-K^0)\gamma^{1/2}d^3x, \tag{5}$$

where γ_{ij} is the metric induced on ∂M, K is the mean extrinsic curvature of ∂M,[6] and K^0 is a constant chosen to make the action of hot flat space with the given boundary ∂M equal to zero.[4] The action (5), when evaluated for the metric (1), can be varied with respect to r, U, and V and one obtains the correct Einstein equations for (1).

In performing a path integral based on the action (5), one wishes to enforce the constraints. We choose to solve the constraints constructed from the metrics (1) explicitly and incorporate the results directly into the action (5), producing in this way a "reduced action" I_*. A path integral using I_* will then involve only those histories that stay on the constraint hypersurface of the gravitational phase space.

For metrics of the form (1), the momentum constraints for τ=const slices are trivially satisfied. The Hamiltonian constraint is

$$(r^2r')^{-1}[r(\hat{V}-1)]'=0, \tag{6}$$

with solution $\hat{V}=1-Cr^{-1}$. The "constant of integration" C is identified as $r(0)=r_+$ from the previously noted requirement $\chi=2$. The parameter r_+ indicates the size of the horizon of the black hole. (For example, $r_+=2GM$ for a Schwarzschild black hole.) Because r_+ arises in solving the Hamiltonian constraint, it can easily be written as a functional of the three-geometry on τ=const slices.

The metric (1) is now reduced to the form

$$ds^2=U(\tau,y)d\tau^2 + (1-r_+/r)^{-1}(r')^2dy^2+r^2(y)d\Omega^2, \tag{7}$$

in which radial gauge invariance (i.e., an essentially arbitrary relation between y and r) is manifest. A point of particular interest is that $U=(\text{lapse function})^2$ is an arbitrary positive function of τ and y, periodic in τ, except that it must take the fixed boundary value U_B at $y=1$ and that its behavior near the origin is fixed, as we infer from (3) and from the form of $\hat{V}$. Near $y=0$, or $r=r_+$, we can write

$$U\xrightarrow[y\to 0]{} 4r_+^2[(1-r_+/r)+a(1-r_+/r)^2+\cdots], \tag{8}$$

where a is a constant. From (7) and (8), a computation of the Einstein tensor shows that the only nonzero components in the limit $y\to 0$ are $G_\theta^\theta=G_\phi^\phi=3a(4r_+^2)^{-1}$, which we can regard as representing quantum (tangential) stresses within the surface of the black hole.

The reduced action I_* is obtained from (5) and (7). Because the Hamiltonian constraint has been eliminated, the essentially arbitrary function U does not enter I_* explicitly. Recall that $K=-g^{-1/2}(g^{1/2}V^{1/2})'$ refers to the hypersurface $y=1$ and that $K^0=-2r_B^{-1}$. The result has the remarkable form

$$I_*=(\beta E-\pi r_+^2/r_P^2)\hbar, \tag{9}$$

where[3] $E=G^{-1}r_B[1-\hat{V}^{1/2}(1)]$ is the energy of a geometry with gravitational radius r_+ and the second term contains the corresponding entropy $S_{\rm BH}$. Thus the reduced action does not have the form "βE" that one would expect for a single state of energy E. The entropy term reveals the presence of a large number of degrees of freedom associated with any value of r_+. Arguments in which $S_{\rm BH}$ is obtained as the logarithm of the number of quantum states accessible in building a black hole,[1] or as the logarithm of the density of states,[5] have been given for black holes that satisfy the complete set of Einstein equations. Here, of course, only the constraint equations have been used. Furthermore, in (9) β, r_B, and r_+ are independent variables, while for the Schwarzschild solution they are necessarily related by the Hawking temperature formula.

If we divide (9) by β, we obtain the "generalized free energy" posited on physical grounds in Ref. 3. Here, we have derived that result from the action evaluated on the constraint hypersurface of the gravitational phase space. One finds that if $r_B\geq(3\sqrt{3}/8\pi)\beta\hbar$, I_* has two stationary points with respect to variation of r_+, which is the only remaining degree of freedom, and at both of them the Hawking temperature formula $\hbar\beta_{\rm H}(r_B)=4\pi r_+\times(1-r_+/r_B)^{1/2}$ holds. One stationary point corresponds to thermodynamically unstable equilibrium, while the other corresponds to equilibrium that is locally stable. In Ref. 3, these stationary points were tacitly identified with the unstable and stable Schwarzschild solutions, respectively, but the result is actually more general. Each stationary point represents an infinite number of equilibrium spacetimes in which Hawking's temperature formula holds, but which are all foliated differently because of the freedom in choosing the lapse function. Only *one* such choice of the lapse at each stationary point would reproduce a Schwarzschild solution, but such a choice is in no way singled out by the reduced action. This wildness in the time slicing expresses strongly that the absence of order in gravitational thermal equilibrium is more radical than that of ordinary thermal equilibrium. In the later case, equilibrium is thought of as occurring *in* spacetime. In the present case, the concept of spacetime is merely heuristic.

If $r_B=(27/32\pi)\beta\hbar$, the value of I_* at the locally stable stationary point vanishes, a fact that will be of great interest below. If $r_B<(3\sqrt{3}/8\pi)\beta\hbar$, then I_* has no stationary points. In Fig. 1, we see that while I_* is not always positive, it is bounded below for every finite choice of β and r_B.

We now examine the partition function, which is

$$Z=\sum_i e^{-\beta E_i}=\sum_E g(E)e^{-\beta E}=\int e^{-\beta E}dN(E), \tag{10}$$

where i labels states and $g(E)$ denotes the multiplicity of energy levels E. In the last equality we have assumed a continuous energy spectrum with measure dN in energy space. In accord with the semiclassical limit of the discussions in Refs. 1 and 5, we take $dN(E)$

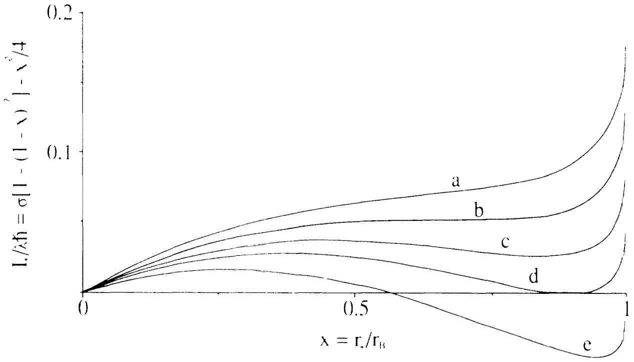

FIG. 1. The reduced action as a function of $x = r_+/r_B$ for several values of $\sigma = \beta\hbar/4\pi r_B$. Curve a ($\sigma = 0.433$) has no stationary points. Curve b ($\sigma = 2/3\sqrt{3} = 0.385$) has two coincident stationary points, while curve c ($\sigma = 0.337$) has a distinct local minimum. For these three curves, the global minimum at $x = 0$ dominates the partition function. Curve d ($\sigma = \frac{8}{27} = 0.296$) has a local minimum with zero action. For curve e ($\sigma = 0.232$), the local minimum dominates the partition function.

$\simeq d[\exp S_{\rm BH}(E)]$. The integral in the last part of (10) is then over $\exp[-\beta E + S_{\rm BH}(E)]$ with measure $dS_{\rm BH}$. Now we observe that, precisely because of the form (9) of the reduced action for any of the geometries (7) in the black-hole topological sector, we are able to write (10) as

$$Z = \int \exp(-I_*/\hbar)\, dS_{\rm BH}. \tag{11}$$

The partition function is expressed as a Euclidean functional integral with measure $\mu = S_{\rm BH} = \pi r_+^2/r_{\rm P}^2$ determined by the metric only at the "center" point of the manifold, i.e., at $y=0$ where $r(0) = r_+$, so that the functional integral reduces to a single ordinary integral. We write (11) in terms of the dimensionless variables $\lambda = 4\pi r_B^2/r_{\rm P}^2$, $\sigma = \beta\hbar/4\pi r_B$, and $x = r_+/r_B$. Then $S_{\rm BH} = \frac{1}{4}\lambda x^2$ and we obtain

$$Z = \tfrac{1}{2}\lambda \int_0^1 dx\, x \exp(-\lambda\{\sigma[1-(1-x)^{1/2}] - \tfrac{1}{4}x^2\}). \tag{12}$$

From the partition function we can obtain the entropy $S_{\rm GF}$ of the gravitational field in the black-hole sector using $S_{\rm GF} = \ln Z - \sigma(\partial/\partial\sigma)\ln Z$. Whenever $\sigma < \frac{8}{27}$, we find that the stable stationary point x_0 dominates Z. At such a point, $\sigma = x_0(1-x_0)^{1/2}$, $1 \ge x_0 > \frac{8}{9}$, and $I_*(x_0) < 0$. Writing $x = x_0 + w$ and expanding through quadratic order in w, we evaluate Z in terms of x_0 and corrections obtained from the Gaussian integral in w. The result contains quantum corrections to $S_{\rm BH}$ evaluated at x_0, namely,

$$S_{\rm GF} = \tfrac{1}{4}\lambda x_0^2 + \ln(2\pi\lambda)^{1/2} x_0 \left(\frac{1-x_0}{3x_0-2}\right)^{1/2} - 1 - 3\frac{x_0(1-x_0)}{(3x_0-2)^2}. \tag{13}$$

The second term is the most important correction to $S_{\rm BH}(x_0)$ when, as we assume, r_B and β are greater than the corresponding Planck values. In this case $S_{\rm GF} > S_{\rm BH}$ as one might have expected. Qualitatively similar results hold for any nonexponential measure $\mu(r_+)$. However, below we obtain a result particular to the choice $\mu = S_{\rm BH}$.

We now turn to the cases in which Z is *not* dominated by the local minima of I_*. If $\sigma > (2/3\sqrt{3})$, then I_* has no stationary points and points near the origin in Fig. 1 dominate Z. But these points do not describe thermal equilibrium as it is normally understood, precisely because they are not classical stationary points of the action or the free energy F ($F = -\beta^{-1}\ln Z$); in fact, we expect that the correct physics will be described in another topological sector, as we discuss further below.

More interesting are the cases in which local minima of I_* exist but are not negative. These occur when $(3\sqrt{3}/8\pi)\beta\hbar \le r_B \le (27/32\pi)\beta\hbar$, or $2/3\sqrt{3} \ge \sigma \ge \frac{8}{27}$. Despite the continued existence of locally stable equilibria in this range, we will see that in the black-hole sector, Z already becomes dominated by points near the origin—tiny nonclassical black holes surrounded by "quantum geometry"—when the local minimum of I_* becomes slightly positive. To see this, it is convenient to change the integration variable in Z from $x = r_+/r_B$ to the dimensionless energy $\epsilon = GE/r_B = 1-(1-x)^{1/2}$. The positive root is taken so that ϵ has the range [0,1]. We have $\mu = S_{\rm BH} = \frac{1}{4}\lambda\epsilon^2(2-\epsilon)^2$ and

$$Z = \lambda \int_0^1 d\epsilon\, \epsilon(1-\epsilon)(2-\epsilon) \times \exp\{-\lambda[\sigma\epsilon - \tfrac{1}{4}\epsilon^2(2-\epsilon)^2]\}. \tag{14}$$

At a stationary point ϵ_0, one has $\sigma = \epsilon_0(1-\epsilon_0)(2-\epsilon_0)$. The case of interest is when, at its local minimum, I_* is zero, the established reference for flat space; this occurs for $\epsilon_0 = \frac{2}{3}$ and $\sigma = \frac{8}{27}$.[7] To explore a neighborhood of this point, we set $\epsilon_0 = \frac{2}{3} + \delta$, and thus $\sigma = \frac{8}{27} - \frac{2}{3}\delta$, for some very small fixed δ. Then the contribution to Z from a stationary point in this region, denoted $Z_{\rm SP}(\delta)$, can be estimated accurately by reducing (14) to a suitable Gaussian. We find

$$Z_{\rm SP}(\delta) \simeq \tfrac{8}{27}(3\pi\lambda)^{1/2}\exp(\tfrac{4}{9}\lambda\delta). \tag{15}$$

On the other hand, the contribution to Z from points near the origin $\epsilon = 0$ is easily seen to be

$$Z_0 \simeq 2/\lambda\sigma^2 = 8\pi/\beta^2 M_{\rm P}^2, \tag{16}$$

for any σ not close to zero. That Z_0 does not depend on the size of the box seems physically correct because this case deals with black holes for which $r_+ \ll r_{\rm P}$ ($\ll r_B$). That Z is independent of r_B is a consequence of using the measure $\mu = S_{\rm BH}$.

The contributions (15) and (16) are equal when $\delta = \delta_0 \simeq \frac{9}{4}\lambda^{-1}\ln(C/\lambda^{3/2})$, where $C \approx 25$. Taking as an example $r_B = 1.5$ km, we have $\lambda \sim 10^{77}$ and $\delta_0 \sim 10^{-76}$.

A dramatic change in Z as the minimum of I_* passes through zero can be seen by noting that $Z_{\rm SP}(0) \sim \lambda^{1/2}$, $Z_{\rm SP}(\delta_0) \sim \lambda^{-1}$, and $Z_{\rm SP}(2\delta_0) \sim \lambda^{-5/2}$. Therefore, Z rapidly becomes well estimated by Z_0 as σ increases through $\frac{8}{27}$. At the same time, the expectation value of the energy undergoes an enormous change, from $\langle E \rangle_{\rm SP} \cong \frac{2}{3}(r_B/G)$ to $\langle E \rangle_0 \cong 2\beta^{-1} \ll M_{\rm P}$. Another significant feature that holds when $Z \cong Z_0$ can be seen by computing the entropy. We find $S_{\rm GF} \approx -2\ln(\beta M_{\rm P})$, which is negative. This results from the fact that when $\sigma > \frac{8}{27}$, the ensemble-averaged number of stationary-point states is actually smaller than one.

The feature we have just described indicates that for $\sigma > \frac{8}{27}$ a phase transition must occur. This should be a transition to a different topological sector, so that Z_0 calculated above would not actually describe a physical situation, for which the entropy would remain positive. One possibility, already previously discussed,[3] is that the transition occurs to the $\chi = 0$ topology corresponding to a box filled with "gravitons," i.e., to the hot-flat-space sector dressed with gravitational fluctuations for which the "one-loop" action is known to be negative. (We do not consider here one-loop perturbative calculations of the effect of small fluctuations in the black-hole sector.) But there exists at present no proof that hot flat space is dominant among the other available topological sectors.

The description of thermodynamic ensembles appropriate to other black-hole geometries—including those endowed with electric charge[8] or in the presence of a nonvanishing cosmological constant[9]—treated in detail by the methods of this paper, will appear elsewhere.

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[7]The values $\sigma = \frac{8}{27}$ and $I_* = 0$ correspond, for a Schwarzschild black hole of mass M, to $r_B = \frac{9}{8}M$, which is the value of the maximum radius of a static, spherical, fluid "star." It also corresponds to the radius of a very thin Wheeler "geon." (We thank D. Brill for this remark.) Whether these coincidences are significant is presently unknown.

[8]H. W. Braden, B. F. Whiting, and J. W. York, Jr., to be published.

[9]J. D. Brown, B. F. Whiting, and J. W. York, Jr., to be published.