Multigrid Method for the Random-Resistor Problem

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(Received 24 March 1988)

We discuss the problem of solving large linear systems of equations that arise in lattice systems with disorder. Three examples of this kind of problem are (i) computing currents in a random-resistor network, (ii) computing the fermion (quark) propagator in lattice quantum chromodynamics, and (iii) the discrete Schrödinger operator with a random potential (the Anderson model of localization). We show that the algebraic multigrid is a very effective way to compute currents in a random-resistor network. It is likely that similar techniques will apply to the other problems.

PACS numbers: 05.50.+q, 02.70.+d, 64.60.Ak, 66.30.Dn

Large sparse linear systems of equations with disordered coefficients arise in several branches of computational physics, and are notoriously difficult to solve. In solid-state physics, the random-resistor network and the Schrödinger operator in a random potential are models of electrical conduction in composite materials and in impure crystals, respectively.[1] In elementary-particle physics, the Dirac propagator in a random gauge field plays a key role in Monte Carlo studies of hadron masses in quantum chromodynamics[2] and in most algorithms for dynamical fermions.[3,4] Critical slowing down is, in general, more severe for disordered systems than for ordered systems, and many acceleration methods that apply to ordered systems may not apply or may work badly for disordered systems.

Recently, Batrouni, Hansen, and Nelkin[5] reported numerical experiments using Fourier acceleration[4,6] for computing currents in random-resistor networks at percolation threshold. They found that critical slowing down is reduced compared to unaccelerated algorithms but is still severe. This is because the preconditioning operator mimics the *ensemble-averaged* current flow in the lattice, while the current flow in any *particular* realization of the random-resistor network is affected strongly by the local and global topology of interconnections. This reasoning suggests that an improved strategy should take account of the topology of the particular resistor network at hand.

The multigrid method[7,8] is known to be an extraordinarily effective approach for solving large linear systems arising from the discretization of elliptic partial differential equations. Usually the coarse grids and interpolation operators are defined geometrically, e.g., cubical (2×2) blocks with piecewise-constant or piecewise-linear interpolation. This approach, which is suitable for partial differential equations with smooth coefficients, will clearly *not* be appropriate in disordered systems such as the random-resistor network: Just because two sites are close geometrically does not mean that they are close in the topology of the resistor network and hence in voltage. Rather, a successful multigrid algorithm will have to define its coarse grids and interpolation operators in accordance with the connection structure of the particular resistor network.

The algebraic multigrid (AMG)[9] is just such an "adaptive" strategy. Heretofore the AMG codes have been applied to partial differential equations with strong jump discontinuities, but not to problems as singular as the random-resistor network. In this Letter we show that a standard AMG code, AMG1R4,[10] succeeds in eliminating entirely (or almost entirely) the critical slowing down in the two-dimensional random-resistor problem. Generalizations of the AMG algorithm to the Dirac-propagator problem[11,12] are currently under investigation.

Our computer experiments are as follows. We work with a two-dimensional lattice having L sites horizontally and $L+1$ sites vertically. Between neighboring sites we insert a bond (unit resistor) with probability p, all insertions being independent. We set the electric potential to be $\phi=0$ on the bottom row, $\phi=1$ on the top row, and compute ϕ on the interior sites (with periodic boundary conditions on the vertical sides) using Kirchhoff's and Ohm's laws. Two preliminary reductions are applied before solving the resulting system of linear equations. First, we compute the "connected cluster" using a

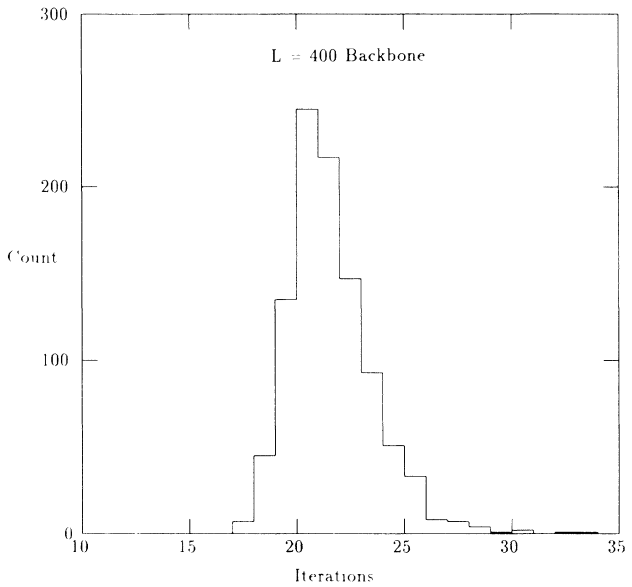

FIG. 1. Histogram of number of iterations needed for convergence on $L=400$ backbones.

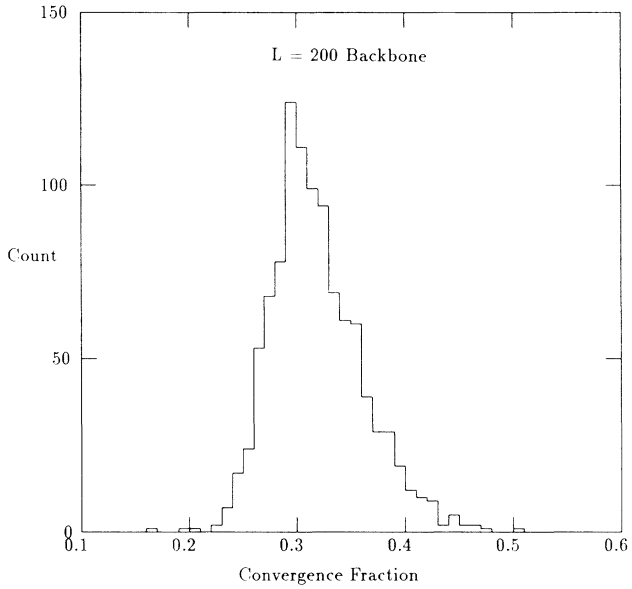

FIG. 3. Histogram of convergence factors for $L=200$ backbone.

modified Hoshen-Kopelman algorithm.[13] A site is in the connected cluster if there is a path of bonds from it to the top and to the bottom. Next we compute the "biconnected cluster" or "current-carrying backbone." A bond is not part of the backbone if for topological reasons it could not possibly carry current, i.e., if it is part of a "dangling end." Tarjan's depth-first-search algorithm[14] finds the backbone in time proportional to the number of sites in the connected cluster. The central-processing-unit time per iteration of the AMG1R4 code is also proportional to the number of unknowns.

We present results for $L=100,200,400$ at $p=p_c=\frac{1}{2}$. We started with an initial guess of $\phi\equiv 0$ in the interior of the lattice and iterated until the error $\|e\| \equiv[\sum_{\text{sites}}(\text{current loss})^2]^{1/2}\leq 10^{-12}$. This corresponds to a reduction in $\|e\|$ by about 13 orders of magnitude. Such high accuracy may be unnecessary for physical applications, but it gives a better understanding of the nu-

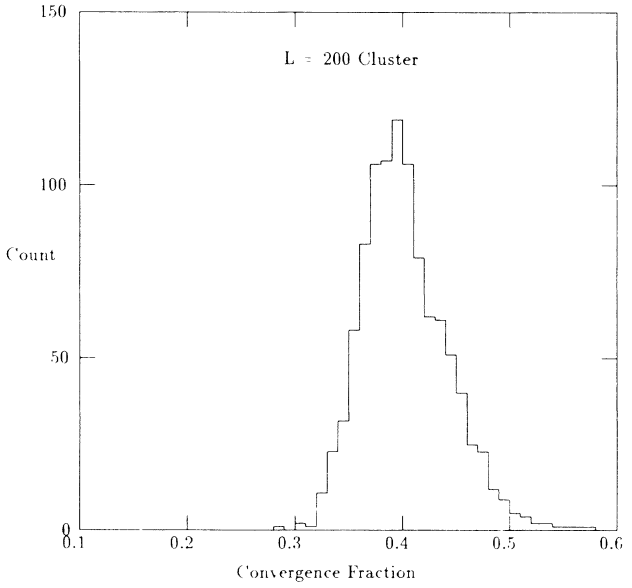

FIG. 2. Histogram of convergence factors for $L=200$ cluster.

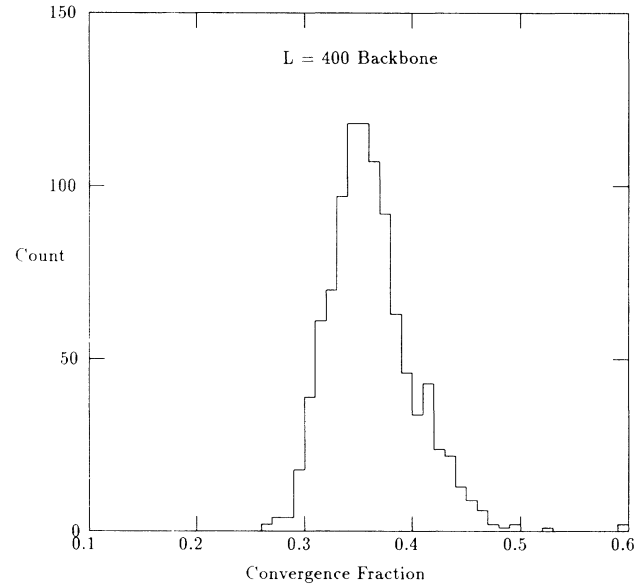

FIG. 4. Histogram of convergence factors for $L=400$ backbone.

TABLE I. Mean convergence factors for algebraic multigrid (AMG) algorithm on two-dimensional random-resistor problem at percolation threshold, computed on either connected cluster or current-carrying backbone. Standard error is shown in parenthesis.

L	Cluster	Backbone
100	0.348 (0.002)	0.264 (0.002)
200	0.403 (0.001)	0.319 (0.001)
400	...	0.362 (0.001)

merical method. Figure 1 is a histogram of the number of iterations needed to reach this criterion, based on 1000 independent realizations. Figures 2, 3, and 4 are histograms of the "convergence factors," i.e., the worst-case factor by which $\|e\|$ is reduced in one multigrid iteration. More precisely, the values shown here are the factors by which the error is reduced in the *last* multigrid iteration. The true (asymptotic) convergence factor is slightly higher than this, but the practically relevant (average) convergence factor is significantly lower. The mean (over realizations) values of this (last iteration) convergence factor are given in Table I.

While the convergence factors increase slightly with L, there is no evidence that they are approaching 1 as $L \to \infty$. We conclude that critical slowing down is completely (or almost completely) absent.

The total time needed to solve the $L = 200$ linear equations on the cluster (backbone) was roughly 5.3 (1.5) min per configuration on a Sun 3/160 work station with floating-point accelerator and 16 Mbytes of memory. Codes for the cluster and backbone computations are available from us.[15] The AMG1R4 code is available from Ruge.[10]

This work was supported in part by National Science Foundation Grants No. PHY-8413569 and No. DMS-8705599. One of the authors (J.G.) was partially supported by a Sloan Foundation Fellowship and by the U.S. Defense Advanced Research Projects Agency.

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