# Universal Singularities in the Integral Quantum Hall Effect

A. M. M. Pruisken
*Department of Physics, Columbia University, New York, New York 10027*
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The transition between localized and extended eigenstates in the integral quantum Hall regime is observable as a scaling phenomenon in the magnetoresistance tensor as a function of temperature and applied magnetic field strength. Field-theoretic studies on the quantum Hall effect are used to derive the scaling functions. The scaling behavior is shown to involve only a single critical exponent ($\mu$) which is the ratio between the inelastic-scattering exponent ($p$) and twice the localization length exponent ($\nu$). The results have recently been confirmed experimentally.

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In a previous Letter,[1] Wei *et al.* reported a novel result on the delocalization phenomenon in the quantum Hall regime. Universal power-law singularities have been observed in the temperature ($T$) dependence of the magnetoresistance data taken from the lowest Landau levels of a low-mobility InGaAs-InP heterostructure. This experiment, which was motivated by the predictions of the present Letter, yields important information on the localization problem[2,3] in the quantum Hall effect (QHE).[4] I will point out that the observed power-law dependence on $T$ is a logical corollary of the scaling theory of the QHE[2,5] and has interesting consequences for field theory[6] and the theory of metallic transport in two dimensions.[7] General scaling relations for the magnetoresistance and conductance will be obtained and expressed in the experimentally accessible variables of $T$ and magnetic field strength $B$ such that the universality statement can be further explored by the experiments and checked for consistency. These predictions are a unique product of the field-theoretic studies on the QHE and cannot be obtained by different means.

Figure 1 gives a standard phenomenological description of the QHE for a single Landau level.[1,3] The appearance of large plateaus with a precisely quantized Hall resistance $\rho_{xy}$ [Fig. 1(a)] is usually explained[8] on the basis of Anderson localized electronic levels in the spectrum of the impurity-broadened Landau level [Fig. 1(b)]. At finite $T$, there is a transition region where these plateaus join smoothly and where the dissipative or parallel resistance $\rho_{xx}$ generally becomes nonzero. In this region, the system behaves like a metal and extended levels near the Fermi energy are necessary in order to account microscopically for the dissipation, as well as the changing $\rho_{xy}$. It is precisely this metallic regime (also regime of "delocalization") which plays the most fundamental role in the theory[2] and which has not been systematically explored in all of the previous experiments. Wei *et al.*[1] discovered a remarkable result in that the maximum slope in the $\rho_{xy}$ curve diverges as a power law in $T$

$$(\partial\rho_{xy}/\partial B)_{\max} \propto T^{-\mu}. \qquad (1)$$

In addition, a half-width for the $\rho_{xx}$ was introduced and this quantity was found to vanish like

$$\Delta_{1/2}B \propto T^{\mu}. \qquad (2)$$

The exponent $\mu = 0.42 \pm 0.04$ was found to be independent of the Landau-level index. In addition, Wei *et al.* reported that the second derivatives with respect to $B$ involve a power law with twice the value of $\mu$ in the exponent.

The fact that the half-width of Eq. (2) and the inverse of Eq. (1) are the same and shrink to zero with $T$ can be considered as the first experimental evidence for the existence of a singular point in the spectrum of each of the Landau levels. This singularity at $T=0$ occurs for a

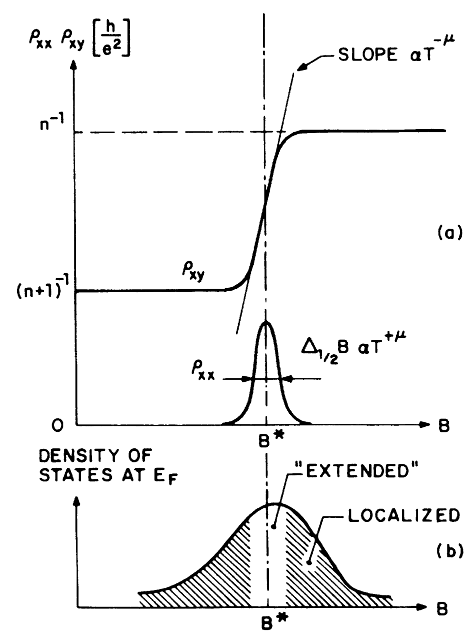


FIG. 1. (a) Sketch of the experimentally measured $\rho_{xx}$ and $\rho_{xy}$ for a single Landau level as a function of the applied magnetic field $B$. (b) Density of levels at the Fermi energy $E_F$ in the impurity-broadened Landau band as $B$ is varied.

specific but nonuniversal value of the magnetic field, say $B^*$. This result, along with the universal power-law behavior in $T$, shows the characteristics of a second-order phase transition as predicted by the renormalization-group theory of the QHE. I will proceed by first quoting several quite distinct and basic features on the basis of which Eqs. (1) and (2) can be understood. A justification then follows as the main objective of the remainder of this Letter.

The experimental observations [Eqs. (1) and (2)] in the metallic regime are a consequence of the result that the magnetoresistance components both depend on the parameters $T$ and $B$ only through the single variable $\kappa$,

$$\rho_{\alpha\beta}(T,B)=r_{\alpha\beta}(\kappa)+(\text{c.s.});\quad \kappa\propto(B-B^*)T^{-\mu}, \tag{3}$$

where $r_{\alpha\beta}$ is a regular function of its argument. The addition (c.s.) summarizes all the corrections to scaling which vanish exponentially or as a power in $T$. The exponent $\mu$ in Eq. (3) appears through the ratio $\mu=p/2\nu$, where $p$ is the inelastic-scattering-length exponent and $\nu$ the localization-length exponent. More specifically, the localization length $\xi$ of the levels near the Fermi energy diverges like a universal power law in $B$,

$$\xi=\xi_0|B-B^*|^{-\nu}. \tag{4}$$

This divergence expresses the fact that at $T=0$ there is a singular point ($B^*$) in the free-electron spectrum of the Landau level. On the other hand, at finite but low enough $T$ such that the thermal broadening of the Fermi-Dirac distribution does not play a significant role, relaxation mechanisms due to inelastic-scattering processes have to be taken into account.[3] These determine a characteristic length (Thouless length[9]) in the problem which behaves like

$$L_{\text{in}}(T)\propto T^{-p/2}, \tag{5}$$

analogous to the theory of ordinary metallic conductors.[9,10] Equations (4) and (5) define a natural scale for the variable $\kappa$ [Eq. (3)]

$$|\kappa|=[L_{\text{in}}(T)/\xi(B)]^{1/\nu}. \tag{6}$$

The condition for metallic behavior can now be expressed by saying $|\kappa|\le 1$, i.e., the mean free path between the inelastic collisions should not exceed the localization length of the levels near the Fermi energy. The statement $|\kappa|\gg 1$, on the other hand, stands for the condition under which the quantum Hall plateaus are observed and will not be considered here.

For a justification and extension of these claims, we make use of the theory of Refs. 2 and 5. This theory is developed for free electrons in a random potential and at $T=0$. The conductance parameters $\sigma_{\alpha\beta}=\rho_{\alpha\beta}/(\rho_{xx}^2+\rho_{xy}^2)$ play the role of two coupled renormalization-group parameters and the effect of changes in the length scale $L$ is illustrated by the flow lines of Fig. 2. The arrows indicate the direction for increasing values of the $L$ and it is

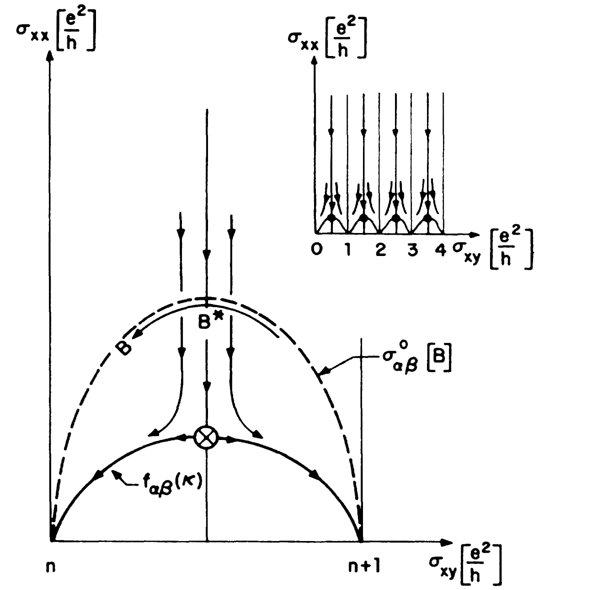


FIG. 2. Translation of Fig. 1(a) into the renormalization-group flow diagram for the conductance parameters. The inset is taken from Refs. 2 and 5. The arrows on the flow lines indicate the direction for increasing length scales $L$. The renormalization-group flow toward the fixed points at integer value for the Hall conductance elucidates the quantum Hall plateaus in Fig. 1(a) as a scaling phenomenon. The unstable fixed point ⊗ controls the universal singularities of the delocalization transition. The parabolic line indicates the semiclassical value $\sigma^0_{\alpha\beta}(B)$ which serves as a starting point for scaling. The $B^*$ corresponds to the singular magnetic field strength at $T=0$ in Fig. 1.

helpful to think of $L$ as being the (effective) sample size. The stable fixed points at $\sigma_{xy}=n,n+1$ in this figure indicate that the quantum Hall plateaus in the experiment [Fig. 1(a)] should be regarded as a universal scaling phenomenon on a macroscopic scale. In contrast, the physics of delocalization is controlled by the unstable fixed point at $\sigma_{xy}=n+\frac{1}{2}$ (⊗ in Fig. 2). I will further clarify this point by making use of the basic principles of the renormalization group along with the topology of the scaling diagram.

The $B$ enters in the scaling diagram of Fig. 2 through the semiclassically computed[2] but nonuniversal values $\sigma^0_{\alpha\beta}(B)$ which are indicated by the parabolic line. These values serve as a starting point for scaling, they are a smooth function of $B$, and there is an associated ultraviolet cutoff $L_0$, determined by the short-distance correlations of the random impurities. The role played by the $\sigma^0_{\alpha\beta}$ is analogous to the role of the kinetic theory (Boltzmann equation) in the conventional theory of localization.[7] The only important point to be made is that there generally exists a single value for the magnetic field, indicated by $B^*$ in Fig. 2, for which the starting point $\sigma^0_{\alpha\beta}$ lies in the domain of attraction (critical domain) of the "delocalization" fixed point ⊗. This $B^*$ will next be identified with the singular magnetic field strength which appeared in the previous discussion. I introduce the scaling variables $\theta$ (relevant) and $\sigma$ (irrelevant),

$$\theta \simeq \sigma_{xy} - n - \tfrac{1}{2}, \quad \sigma \simeq \sigma_{xx} - \sigma_{xx}^*, \tag{7}$$

to denote the linear environment of the unstable fixed point ⊗. Under a renormalization-group transformation $L \to bL$, these variables transform according to $\theta \to b^{y_\theta}\theta$, $\sigma \to b^{-y_\sigma}\sigma$. The localization length transforms as a length such that[2]

$$\xi(\theta,\sigma) = b\xi(b^{y_\theta}\theta, b^{-y_\sigma}\sigma). \tag{8}$$

The generalized homogeneous form of Eq. (8) implies $\xi = |\theta|^{-1/y_\theta} A(|\theta|^{y_\sigma/y_\theta}\sigma)$ which is finite everywhere except along the critical domain $\sigma_{xy} = n + \frac{1}{2}$.[11] Since only the relevant variable matters, one identifies $\theta \propto B - B^*$ from the starting points for scaling (Fig. 2) which then directly leads to Eq. (4). The exponent

$$\nu = y_\theta^{-1} \tag{9}$$

is predicted to be universal.[5]

Next, it is somewhat less obvious how to relate the conductance parameters at large length scales $L$ to the starting points $\sigma_{\alpha\beta}^0(B)$ at $L_0$. The former will be denoted as $\sigma_{\alpha\beta}(L,B)$. It is clear from Fig. 2 that all of the initial points with $B \neq B^*$ will end up very close to the "quantum Hall" fixed points with $\sigma_{xy}$ = integer, if $L$ is large enough (more precisely $L > \xi$). For $L$ finite but large, only a small fraction $\sigma_{\alpha\beta}(L, B \approx B^*)$ does not satisfy this condition and ends up close to the symmetric renormalization trajectory connecting "delocalization" fixed points with the "quantum Hall" fixed points. It is precisely this small fraction which describes the metallic regime and which is of interest to us. The problem is most elegantly formulated by introduction of Wegner's scaling fields[12] which generalize Eq. (7) to the entire conductance plane. These generalized variables $\theta = \theta(\sigma_{\alpha\beta})$, $\sigma = \sigma(\sigma_{\alpha\beta})$ are curvilinear coordinates adapted to the transformation properties which are written above Eq. (8). These variables are formally given by an infinite Taylor series in the $\sigma_{\alpha\beta}$. The series can be inverted such that the conductance parameters can also be expressed as a regular function $\sigma_{\alpha\beta} = F_{\alpha\beta}(\theta,\sigma)$. The simple transformation properties of the generalized scaling variables can next be used to write

$$\sigma_{\alpha\beta}(L,B) = F_{\alpha\beta}((L/L_0)^{y_\theta}\theta_0(B), (L/L_0)^{-y_\sigma}\sigma_0(B)). \tag{10}$$

Here the $\theta_0, \sigma_0$ stand for the translation of starting points $\sigma_{\alpha\beta}^0(B)$ into the generalized Wegner fields. Since both the former and the latter functions are regular functions of their independent variables, one can write down the series

$$\begin{aligned} \theta_0(B) &= a_1(\delta B) + a_2(\delta B)^2 + \cdots, \\ \sigma_0(B) &= b_0 + b_1(\delta B) + b_2(\delta B)^2 + \cdots, \end{aligned} \tag{11}$$

where $\delta B = B - B^*$. Equations (10) and (11) constitute a formal solution of the problem stated at the outset and can be used to discuss the leading singularities as $L \to \infty$. With the introduction of the variable $\kappa \propto L^{y_\theta}\delta B$, then the asymptotic behavior can be written

$$\begin{aligned} &\sigma_{\alpha\beta}(L,B) = f_{\alpha\beta}(\kappa) + O(L^{-y_\theta}\kappa^2, L^{-y_\sigma}), \\ &\kappa \propto L^{y_\theta}(B - B^*), \end{aligned} \tag{12a}$$

which is a function of the single variable $\kappa$ except for the power-law corrections. The $f_{\alpha\beta}(\kappa) \equiv F_{\alpha\beta}(\kappa,0)$ is identified as the symmetric trajectory between the two kinds of fixed points in Fig. 2. The function $f_{\alpha\beta}$ can be obtained as a regular power series in $\kappa$ everywhere except near the quantum Hall fixed points where the condition for the QHE is satisfied, i.e., $L > \xi$. Using the normalization $|\kappa| = [L/\xi(B)]^{1/\nu}$, one expects

$$f_{xx}(\kappa), |f_{xy}(\kappa) - f_{xy}(\infty)| \propto \exp(-|\kappa|^\nu) = \exp(-L/\xi),$$

for $\kappa \gg 1$. The statement of Eq. (12a) can be directly translated into resistances,

$$\rho_{\alpha\beta}(L,B) = r_{\alpha\beta}(\kappa) + O(L^{-y_\theta}\kappa^2, L^{-y_\sigma}), \tag{12b}$$

with $r_{\alpha\beta} = (f^{-1})_{\alpha\beta}$. Equations (9) and (12b) reproduce the statement made in Eq. (3), by substitution of the Thouless length $L_{\text{in}}(T)$ [Eq. (5)] for $L$.

The absolute scale in $T$ for observing the basic result of Eq. (3) is very much dependent on the microscopic details of the randomness as well as the Landau-level index.[1-3] Moreover, in comparing Eq. (3) with the experimental data, it is extremely important to keep in mind that the universality statement concerns the value for the exponent $\mu$. The $B^*$, on the other hand, is nonuniversal. For example, a slight macroscopic inhomogeneity in the electron density across the sample will in general result in slightly different values of $B^*$ in Eq. (3) for the (differently) measured $\rho_{xx}$ and $\rho_{xy}$. Such inhomogeneities do not affect the power-law behavior of Eqs. (1) and (2) but they do complicate the inversion of resistances into conductances. This is one of the reasons why the delocalization fixed points ⊗ could not be observed more directly by plotting the experimental data as $T$-driven flow lines in the $\sigma_{xx}$-$\sigma_{xy}$ conductance plane.[13] Finally, the result of Eq. (3) implies that all the $B$ derivatives of the magnetoresistance data diverge as a power law $T$ with integer multiples of $\mu$ in the exponent. This prediction can be used as a further check in the experiments.

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