Possible Phase Transitions among Calabi-Yau Compactifications

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We study a wide class of complex spaces each of which may be spanned by "internal" degrees of freedom of certain physics models with simple supergravity in (3+1)-dimensional Minkowski space-time. We show that there are connections among these complex spaces suggesting the possibility of phase transitions which would cause drastic changes in the physical observables of such models. The web of superstring models so connected suggests the existence of a "unified" superstring model of which the models studied here are special cases.

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Superstrings propagating in (3+1)-dimensional Minkowski space-time with simple supergravity are described by certain [(1+1)-dimensional] world-sheet field theories[1]—myriads of which are known to exist. A subset of the degrees of freedom of such a world-sheet field theory spans the (3+1)-dimensional Minkowski space-time and we refer to the remaining degrees of freedom as "internal"; the main observables of the model are then typically determined by this internal structure. Some of these models are known to be limiting cases of others (e.g., superstrings on orbifolds[2] and on their blowups[3]) but it is not clear whether such a "connectedness" exists in general.

An appreciable subset of superstring models (i.e., classical vacua of a superstring theory) are described by nonlinear σ models on the world sheet, where even string tree-level results are obtained only in (various) approximation schemes. The internal degrees of freedom are, however, known to span Calabi-Yau manifolds and also give rise to what is recognized as gauge interacting matter.[4] Among superstring models with such internal Calabi-Yau manifolds we here focus on those which correspond to world-sheet field theories with (2,2) supersymmetry, with the understanding that our results have implications in a wider scope.

A superstring model S with an internal Calabi-Yau manifold $\mathcal{M}$ depends on parameter θ which parametrize the complex structure of $\mathcal{M}$. For the theories under consideration, it has been proved by Tian[5] that the space Θ of the parameters θ is a complex manifold of dimension $b_{2,1}(\mathcal{M})$ whose tangent space at $\mathcal{M}$ may be identified with $H_{\bar\partial}^{(2,1)}(\mathcal{M})$. In addition, S also depends on the choice of the Kähler class of $\mathcal{M}$, $J(\mathcal{M})$, as a real (i.e., self-conjugate) positive element of $H_{\bar\partial}^{(1,1)}(\mathcal{M})$. The possible choices for $J(\mathcal{M})$ form a cone $\Xi(\mathcal{M})$ spanned by $b_{1,1}$ real parameters ξ.

Corresponding to $\phi\in H_{\bar\partial}^{(2,1)}(\mathcal{M})$ and $\psi\in H_{\bar\partial}^{(1,1)}(\mathcal{M})$, there exist $b_{2,1}+b_{1,1}$ gauge-invariant massless superfields Φ and Ψ, the potential of which is derived to vanish to all finite orders in string perturbation.[6] Being related to the structure on $\mathcal{M}$, these modes are called "moduli," but to emphasize the fact that they are *tangential* to the actual moduli space of S, we shall refer to Φ and Ψ as "moduli fields." In so-called heterotic models, $b_{2,1}$ and $b_{1,1}$ also count massless standard-model families of chiral superfields and the number of mirror families, respectively.

In this note we wish to clarify and illustrate the fact that, for a huge number of Calabi-Yau manifolds[7,8]—of different homotopy types and in particular many different $b_{2,1}$ and $b_{1,1}$—moduli spaces are connected in the sense that they can be identified with (partial) "boundaries" of one another.[9] Since these moduli parametrize both Calabi-Yau manifolds and corresponding superstring models, this connectedness suggests phase transitions and seems to offer a framework for analysis of this phenomenon; a detailed and more complete study is included in a separate project.[10]

We start with an example of perhaps the simplest of these transformations, related to "contraction," first discussed in Ref. 8. Consider a Calabi-Yau subspace of $CP^4\times CP^1$, the subspace of simultaneous solutions to

$$\begin{aligned} A_1(z;w)&=\sum\theta^{(1)}_{abci}z^az^bz^cw^i=0,\quad \theta^{(1)}_{abci}\in C,\\ A_2(z;w)&=\sum\theta^{(2)}_{dej}z^dz^ew^j=0,\quad \theta^{(2)}_{dej}\in C, \end{aligned}\tag{1}$$

where z^a and w^i are homogeneous coordinates of CP^4 and CP^1, respectively.

The space of the coefficients $\theta^{(1)}$ and $\theta^{(2)}$, $\Theta_+(\mathcal{M})$, is a considerably redundant deformation space for (possibly singular) varieties $\mathcal{M}$, which are said to *belong* ($\in$) to the *configuration matrix*

$$\left[\begin{matrix}4\\1\end{matrix}\middle\|\begin{matrix}3&2\\1&1\end{matrix}\right],$$

in our notation, where the first column represents $CP^4 \times CP^1$ while the latter two display the degrees of homogeneity of $A_i(z;w)$. For a generic choice of $\theta^{(1)}$'s and $\theta^{(2)}$'s (i.e., for almost all choices), $\mathcal{M}$ is nonsingular and has[11] $b_{1,1}=2$ and $b_{2,1}=66$.

We can easily construct a sharper deformation space $\Theta(\mathcal{M})$ by identifying points of $\Theta_+(\mathcal{M})$ under the following equivalences:

$$\begin{aligned} &A_i(z;w) \cong \lambda_i \cdot A_i(z;w), \\ &A_1(z;w) \cong A_1(z;w) + \left[\sum \lambda_a z^a\right] \cdot A_2(z;w), \\ &A_i(z;w) \cong A_i(z';w) \cong A_i(z;w'), \\ &z'^a = \gamma_{(4)}{}^a{}_b z^b, \quad w'^i = \gamma_{(1)}{}^i{}_j w^j, \end{aligned} \tag{2}$$

where $\gamma_{(4)}{}^a{}_b$ and $\gamma_{(1)}{}^i{}_j$ are linear reparametrizations of CP^4 and CP^1, respectively. Note that $\dim\Theta_+(\mathcal{M})=100$ but $\dim\Theta(\mathcal{M})=100-2-5-24-3=66=b_{2,1}$.

Note next that the two homogeneous equations $A_i(z;w)=0$ on the two homogeneous coordinates w^1 and w^2 *imply* the determinant equation

$$0=\Delta(z) \equiv [A_{1,1}(z)A_{2,2}(z) - A_{1,2}(z)A_{2,1}(z)],$$
$$A_{i,j}(z) \equiv [\partial A_i(z;w)/\partial w^j],$$

since both w^1 and w^2 cannot vanish on CP^1. Now, $\Delta(z)=0$ is a (nongeneric) quintic polynomial constraint on CP^4 and itself defines a *singular* variety $\mathcal{M}^\# \in [4\|5]$. This is seen from the fact that both $\Delta(z)$ and

$$\partial\Delta(z) = \{[\partial A_{1,1}(z)]A_{2,2}(z) - [\partial A_{1,2}(z)]A_{2,1}(z) + (1 \leftrightarrow 2)\}$$

vanish when all four $A_{i,j}(z)$ do. These are two cubic and two quadric equations in CP^4, which generically have solutions—36 isolated points. These are the only singularities of $\mathcal{M}^\#$ and each is, locally, a vertex of a cone over $CP^1 \times CP^1$, obtained by "contracting" a $CP^1 \in \mathcal{M}$, and we shall refer to $\wp: \mathcal{M} \to \mathcal{M}^\#$ as a *contraction.* The reversed transition consists of a *small resolution*[12] of each singularity of $\mathcal{M}^\#$ into a $CP^1 \subset \mathcal{M}$. Note that *this is not blowing up*; the existence of small resolutions is an algebraic accident for $\dim\mathcal{M}=3$.

It is the same 100 $\theta^{(1)}$'s and $\theta^{(2)}$'s that parametrize $\Delta(z)$ and therefore $\mathcal{M}^\#$, so that $\Theta_+(\mathcal{M})$ can also be taken to be a deformation space $\Theta_+(\mathcal{M}^\#)$ for $\mathcal{M}^\#$. Note that each of the equivalence relations (2) of $\Theta_+(\mathcal{M})$ induces an equivalence relation for $\Theta_+(\mathcal{M}^\#)$ so that in fact $\Theta(\mathcal{M})$ is also a deformation space for $\mathcal{M}^\#$.

Finally, we observe that the equation

$$\Delta(z)+\delta(z)=0, \tag{3}$$
$$\delta(z) = \sum \theta^{(3)}_{abcde} z^a z^b z^c z^d z^e, \quad \theta^{(3)}_{abcde} \in C,$$

with $\delta(z)$ a generic perturbation of $\Delta(z)$ transverse (i.e., normal in some suitable metric) to the space of polynomials parametrized by $\Theta_+(\mathcal{M})$, describes a nonsingular Calabi-Yau threefold $\mathcal{M}^{\mathrm{b}}$ with $b_{1,1}=1$ and $b_{2,1}=101$. The parameters $\theta^{(1)}$, $\theta^{(2)}$ [in $\Delta(z)$], and $\theta^{(3)}$ [in $\delta(z)$] span a considerably redundant deformation space, $\Theta_+[4\|5]$, for possibly singular quintic hypersurfaces in CP^4 of the form (3). Quite obviously, $\Theta_+[4\|5]\,|_{\theta^{(3)}=0} = \Theta_+(\mathcal{M}^\#)$.

The transition $\delta^*: \mathcal{M}^\# \to \mathcal{M}^{\mathrm{b}}$ is clearly a deformation, and the sequence of transitions

$$\mathcal{M} \xrightarrow{\wp} \mathcal{M}^\# \xrightarrow{\delta^*} \mathcal{M}^{\mathrm{b}}$$

induces the identifications

$$\Theta_+(\mathcal{M}) = \Theta_+(\mathcal{M}^\#) = \Theta_+[4\|5]\,|_{\theta^{(3)}=0}$$

(see Fig. 1).

The present understanding of superstring theory does not supply techniques for the analysis of a possible superstring model $S^\#$, with internal $\mathcal{M}^\#$. It would certainly be desirable to check if such a superstring model would be consistent and if so, compute the observables and compare with superstring models constructed by different techniques. This is, however, beyond our present scope and fortunately irrelevant for the sequel.

As a result of the identification $\Theta_+(\mathcal{M}) = \Theta_+(\mathcal{M}^\#)$, we may, and in fact will, regard $\Theta_+(\mathcal{M}) = \Theta_+[4\|5]\,|_{\theta^{(3)}=0}$. Note now that

$$\lim_{\theta^{(3)} \to 0} \Theta_+(\mathcal{M}^{\mathrm{b}}) = \Theta_+[4\|5]\,|_{\theta^{(3)}=0}.$$

Moreover, since a nodal variety such as $\mathcal{M}^\#$ has only a finite set of distinct small resolutions, it follows that the contraction $\wp$ identifies the space $\Theta(\mathcal{M})$ of Calabi-Yau manifolds defined by Eq. (1) with a (possibly trivial) connected covering space of a singular subset of the space $\Theta[4\|5]$ of (possibly singular) quintic hypersurfaces in CP^4, i.e., with a covering space of part of the boundary of the space $\Theta(\mathcal{M}^{\mathrm{b}})$ of nonsingular quintic hypersurfaces in CP^4.

We also note that the Kähler class of $\mathcal{M}$, $J(\mathcal{M})$, is parametrized by $\Xi(\mathcal{M})$, a positive cone of two parameters:

$$J(\mathcal{M}) = \xi_1 J(CP^4) + \xi_2 J(CP^1)$$

while $J(\mathcal{M}^{\mathrm{b}}) = \xi_1 J(CP^4)$ [with, e.g., the Fubini-Study

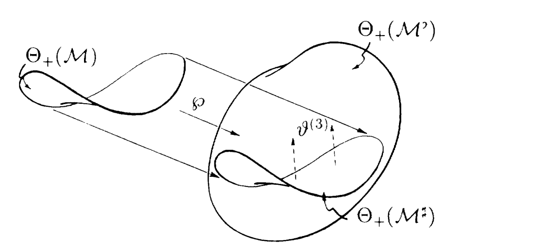

FIG. 1. Relation of moduli spaces $\Theta_+(\mathcal{M})$, $\Theta_+(\mathcal{M}^\#)$, and $\Theta_+(\mathcal{M}^{\mathrm{b}}) \subset \Theta_+[4\|5]$.

metric on CP^n, the Kähler class is given by $J(CP^n)=\frac{1}{2}i\partial\bar\partial\ln(1+\sum z\cdot\bar z)$]. It makes sense to define $\Xi(\mathcal{M}^{\#})$ to be $\lim_{\xi_2\to 0}\Xi(\mathcal{M})$. Then the model $S^{\#}$ is on the boundary of both S and S^{b} but corresponds in the first case to limiting values of the Kähler class and in the second to limiting values of the moduli of the complex structure.

Several remarks are now in order.

(i) The transition $\mathcal{M}\to\mathcal{M}^{\mathrm{b}}$ between two Calabi-Yau manifolds strongly suggests a phase transition between corresponding superstring models $S\to S^{\mathrm{b}}$. This happens at the critical subspace of the moduli space $\Theta[4\|5]$, as $\theta^{(3)}\to 0$.

(ii) Next, note that the polynomial deformation method[11] represents $\phi\in H_{\bar\partial}^{(2,1)}(\mathcal{M})$, $\phi^{\mathrm{b}}\in H_{\bar\partial}^{(2,1)}(\mathcal{M}^{\mathrm{b}})$ correctly and therefore also the moduli fields Φ and Φ^{b} of S and S^{b}, respectively.

(iii) Comparison of the defining equations suggests that the Φ moduli fields would have to undergo a *nonlinear* transformation in a $S\to S^{\mathrm{b}}$ phase transition:

$$\{\Phi_{abc},\Phi_{de}\}\to\{\Phi^{\mathrm{b}}_{abcde}=\Phi_{(abc}\cdot\Phi_{de)}+\Phi^{\mathrm{b}}_{abcde}(\theta^{(3)})\},$$

where $\Phi^{\mathrm{b}}_{abcde}(\theta^{(3)}=0)=0$.

(iv) On the other hand, the Kähler-class modulus field which corresponds to $J(CP^1)$ would have to vanish from the massless spectrum *away from* $\theta^{(3)}=0$.

(v) Furthermore, the number of matter chiral superfields would change as well in such a transition, from 66 **27**'s and two **27***'s of S to 101 **27**'s and only one **27*** of S^{b}. Note, however, that the gauge group would remain intact.

(vi) Finally, we remark on differences from the transitions induced by the blowing up of the singularities of toroidal orbifolds: There the number of moduli fields and matter fields remains the same, but the enhanced gauge symmetry [$\subseteq$ SU(3)] breaks and becomes (part of) the SU(3) holonomy of the Calabi-Yau manifold into which the orbifold has been blown up. In the case of contractions (and reversely, small resolutions of singular deformations) the geometry of the transition is completely different and the gauge symmetry remains intact while the number of moduli fields and matter fields changes drastically.

The transition illustrated above is easily generalized into

$$\mathcal{M}\in\left[\begin{matrix}X\\ n\end{matrix}\middle\|\begin{matrix}A_1&\dots&A_{n+1}\\ 1&\dots&1\end{matrix}\right]\to\mathcal{M}^{\mathrm{b}}\in[X\|(A_1+\cdots+A_{n+1})],\tag{4}$$

where X stands for any configuration matrix representing a fourfold and A_i's are any nonvanishing column vectors of integers such that $\mathcal{M}$ (and consequently a generic $\mathcal{M}^{\mathrm{b}}$) is a Calabi-Yau manifold (see Refs. 7, 8, or 11 for details of this condition). Another generalization is provided[9] by

$$\mathcal{M}\in\left[\begin{matrix}X\\ n\end{matrix}\middle\|\begin{matrix}Q&L_2&\dots&L_{n+1}\\ 2&1&\dots&1\end{matrix}\right]\to\tilde{\mathcal{M}}\in[[X]](2Q+2L_2+\cdots+2L_{n+1}),$$

where X is any configuration matrix representing an almost-Fano threefold and Q and L_i are any nonvanishing column vectors of integers such that $\mathcal{M}$ is a Calabi-Yau manifold. We denote by $[[X]](B)$ the double covering of X branched over a nonsingular surface that belongs to $[X\|B]$; it follows that $\tilde{\mathcal{M}}$ is a Calabi-Yau manifold.

These contractions have an important property in common: Certain complex lines $l\approx CP^1\subset\mathcal{M}$ are contracted to singular points of $\mathcal{M}^{\#}$ which is then deformed into $\mathcal{M}^{\mathrm{b}}$; this point of view is manifestly independent of particular constructions. In fact, Reid[13] conjectures that a large class of moduli spaces of threefolds with trivial canonical bundle (these include Calabi-Yau manifolds as nonsingular Kähler cases) can be identified with a class of subspaces of a universal moduli space.

While the existence of such a universal moduli space is subject to a conjecture, for Calabi-Yau manifolds constructed as in Refs. 7 and 8, we can prove that their moduli spaces are all connected to one another by iterations of the transition (4), e.g.,

$$\left[\begin{matrix}1\\1\\1\\1\end{matrix}\middle\|\begin{matrix}2\\2\\2\\2\end{matrix}\right]\longleftarrow\left[\begin{matrix}4\\1\\1\\1\\1\end{matrix}\middle\|\begin{matrix}1&1&1&1&1\\1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\1&0&0&0&1\end{matrix}\right]\longrightarrow\left[\begin{matrix}4\\1\\1\\1\end{matrix}\middle\|\begin{matrix}2&1&1&1\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{matrix}\right]\longrightarrow\cdots\longrightarrow[4\|5].$$

In fact, it is straightforward that any configuration matrix is connected, through an analogous sequence of transitions, to the leftmost one above; thus, all configuration matrices are connected. Each arrow in the web of all such connections represents an identification of one moduli space with a covering space of a critical subset of another (see Fig. 1) and also corresponds to a phase transition among corresponding superstring models. We defer to a future study the questions of whether these coverings are trivial and of how Reid's conjecture should be reformulated in the event that they are not.

From the point of view of analyzing superstring models with internal Calabi-Yau manifolds, Reid's universal moduli space could be a "universal parameter space" for a "unified" superstring model (unifying corresponding vacua of a superstring theory). Regardless of the status of Reid's conjecture, our analysis[9] provides connections and suggests phase transitions among an appreciable subset of superstring vacua, or for that matter any other models having internal Calabi-Yau manifolds as discussed here.

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