Knot Theory and Quantum Gravity

Carlo Rovelli
Sezione di Roma, Instituto Nazionale di Fisica Nucleare, University of Rome, Rome, Italy

and

Lee Smolin[a]
Department of Physics, Yale University, New Haven, Connecticut 06520
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A new representation for quantum general relativity is described, which is defined in terms of functionals of sets of loops in three-space. In this representation exact solutions of the quantum constraints may be obtained. This result is related to the simplification of the constraints in Ashtekar's new formalism. We give in closed form the general solution of the diffeomorphism constraints and a large class of solutions of the full set of constraints. These are classified by the knot and link classes of the spatial three-manifold.

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Despite the failure of standard perturbative quantizations, many people have argued that quantum general relativity may still exist because strong-coupling effects at short distances contradict the assumption, which underlies perturbation theory, that quantum geometry may be understood in terms of small fluctuations around a classical background spacetime.[1] One approach to the investigation of this hypothesis is canonical quantization, in which the splitting of the metric into a classical background part and a fluctuating quantum part is not made.[2] In the canonical formulation of general relativity, for the case of closed space Σ, the Hamiltonian is weakly vanishing and in the quantum theory the dynamics is expressed by the quantum constraint equations.

In this Letter we describe a new representation of canonical quantum general relativity, called the loop representation, in which exact, nonperturbative, solutions to the constraint equations may be obtained.[3] In particular, we describe here the following results.

(1) The entire space of states annihilated by the spatial diffeomorphism constraints $D_a(x)$ is found in terms of an explicit countable basis. The elements of this basis are in one-to-one correspondence with the generalized link classes of the 3D manifold Σ. These are the equivalence classes, under $\mathrm{Diff}(\Sigma)$, the identity-connected component of the diffeomorphism group of Σ, of sets of piecewise differentiable loops in Σ.

(2) Among these states are some which are also annihilated by the Hamiltonian constraint $\mathcal{C}(x)$, and are thus exact physical quantum states of the gravitational field. Included in these is a sector whose basis is in one-to-one correspondence with the subset of the generalized link classes of Σ which are based on sets of smooth, nonintersection loops. These are the well studied ordinary link classes, whose classification is the subject of knot theory.[4]

The loop representation is a development of Ashtekar's reformulation of general relativity[5] and is motivated by the discovery[6] of a set of solutions of the Wheeler-DeWitt equation related to loops. In Ref. 7 it was first introduced by means of a functional transform from the self-dual representation.[5] Here, following Isham's ideas,[8] we define directly the loop representation as the quantization of a suitable Poisson algebra of nonlocal classical observables.

We proceed by describing the loop representation, we then explain why it is a quantization of general relativity, and, finally, we describe how the solutions are found.

Let Σ be a compact three-manifold, of arbitrary topology, without metric or connection structure. Let $\mathcal{L}_\Sigma$ be the space of piecewise differentiable, closed, parametrized, nondegenerate curves in Σ (called in what follows, loops, and denoted by greek letters $\gamma, \eta, \ldots$) and let $\mathcal{M}_\Sigma$ be the space of the (unordered) set of elements of $\mathcal{L}_\Sigma$ (called multiple loops and denoted $\{\gamma\}, \{\eta\}, \ldots$). Let $\mathcal{S}$ be the space of complex-valued functions $\mathcal{A}[\{\gamma\}]$ on $\mathcal{M}_\Sigma$ which (1) are invariant under reparametrization and inversion of the loops, and (2) satisfy, for any γ and η with a common base point, the equation $\mathcal{A}[\gamma\#\eta]+\mathcal{A}[\gamma^{-1}\#\eta]=\mathcal{A}[\{\gamma,\eta\}]$, where $\gamma^{-1}(s)\equiv\gamma(1-s)$ ($s\in[0,1]$ is the loop parameter) and $\gamma\#\eta$ is the loop made by going once round γ and then once round η before closing. (As in spin network formalism,[9] this is an implementation in the loop space of the fundamental two-spinor identity, $\delta_A^B\delta_C^D-\delta_A^D\delta_C^B=\epsilon_{AC}\epsilon^{BD}$.)

On this space $\mathcal{S}$ there exists an algebra of regulated linear operators which is a representation of a complete, observable algebra for general relativity. The algebra, called the $\tilde{T}$ algebra, is graded by the nonnegative integers. The zero elements are defined for every loop γ by

$$\tilde{T}^0[\gamma]\mathcal{A}[\{\eta\}]=\mathcal{A}[\gamma\cup\{\eta\}].$$

These are a kind of lowering operator. For $n\geq 1$ the operators are denoted $\tilde{T}_\epsilon^{n,a_1,\ldots,a_n}[\gamma](s_1,\ldots,s_n)$ (a_i

$=1,2,3$, $s_i+\epsilon<s_{i+1}$, $\epsilon>0$) and are associated with operations in which new set of loops are made from old ones by breaking and joining them at points of intersection. $\tilde{T}_\epsilon^{n,a_1,\dots,a_n}[\gamma](s_1,\dots,s_n)\ \mathcal{A}[\eta]$ will be zero unless η intersects γ at all the n points $\gamma(s_i)$. In this case we consider the 2^n loops (or multiple loops), denoted $(\gamma\#\eta)_p$, $p=1,\dots,n^2$, obtained by breaking and rejoining the two loops in each of the n intersections ($\times\rightarrow \genfrac{}{}{0pt}{}{\vee}{\wedge}$, $><$). Then

$$\tilde{T}_\epsilon^{n,a_1,\dots,a_n}[\gamma](s_1,\dots,s_n)\mathcal{A}[\eta]\equiv\hbar^n\sum_p(-1)^{|p|}\Delta^{n,a_1,\dots,a_n}[\gamma,\eta](s_1,\dots,s_n)\mathcal{A}[(\gamma\#\eta)_p].$$

$|p|$ is the number of the segments between two intersections on which an arbitrary original orientation of the two loops has to be reversed in order to obtain a coherent orientation of $(\gamma\#\eta)_p$. The coefficients Δ are defined by

$$\Delta^{n,a_1,\dots,a_n}[\gamma,\eta](s_1,\dots,s_n)\equiv\int dt_1\cdots dt_n\delta^3(\gamma(s_1),\eta(t_1))\dot{\eta}^{a_1}(t_1)\cdots\delta^3(\gamma(s_n),\eta(t_n))\dot{\eta}^{a_n}(t_n). \tag{1}$$

These are distributional constants, which enforce the requirement that the action of the operator be zero unless η intersects γ at the points $\gamma(s_i)$. The definition of the operators incorporates a regularization: For every ϵ the set of the operators T_ϵ^n is given by the ones in which the points s_i are at least a distance ϵ apart in terms of the parameter along γ. The extension of these definitions to operators depending on multiple loops is straightforward.

It can be demonstrated that for each fixed $\epsilon>0$, the $\tilde{T}_\epsilon^n$'s form a closed commutator algebra, which is of the form, for $n\geq m$,

$$[\tilde{T}_\epsilon^n,\tilde{T}_\epsilon^m]\sim\hbar\Delta^1\tilde{T}^{n+m-1}+\hbar^2\Delta^2\tilde{T}_\epsilon^{n+m-2}+\cdots+\hbar^n\Delta^n\tilde{T}_\epsilon^m,$$

where the terms in $\tilde{T}^{n+m-p}$ involve loops which are formed by breaking and joining γ and η at p intersection points. Thus, the Δ^n defined by Eq. (1) are structure constants of the algebra of the $\tilde{T}_\epsilon^n$. The algebra admits a graphical notation[3] which allows direct computations, in terms of breaking and joining of loops at intersection points. The distributional singularities present in the definition of the $\tilde{T}_n$ and in their algebra may be eliminated[3] by averaging over suitable finite-dimensional spaces of test functions. The ϵ regularization ensures that there are no coincident distributional singularities.

Now, what has all of this to do with quantum gravity? It may be shown that the algebra $\tilde{\mathcal{T}}$ is, if we divide the commutators by $i\hbar$ and take the limit $\hbar\rightarrow0$, isomorphic to the Poisson-bracket algebra of a certain set of observables in the classical phase space of general relativity. This algebra, which we will denote $\mathcal{T}$, without the tilde, is complete, in the sense that any observable is either in it or may be expressed as a limit of sequences of its elements. The T^0's are the traces of the holonomy of Ashtekar's connection,[5] $A_a(x)$,

$$T^0[\gamma]=\mathrm{Tr}P\exp\left(\oint_\gamma A\right),$$

where P means path ordered. The T^0's, being the holonomy of an SU(2) connection, satisfy certain algebraic conditions at the intersecting points on the loops[6]; the quantum $\tilde{T}^0$ operators satisfy the same condition thanks to the spinor condition, above, on the loop functionals.

The higher T^n are defined by our inserting, into the loops, n insertions of the conjugate variable $\tilde{\sigma}^a(x)$. If

$$U_\gamma(s,t)\equiv P\exp\left(\int_s^t A(\gamma(s))_a\dot{\gamma}^a ds\right),$$

$$T_c^{n,a_1,\dots,a_n}[\gamma](s_1,\dots,s_n)\equiv\mathrm{Tr}[U_\gamma(s_n,s_1)\tilde{\sigma}^{a_1}(\gamma(s_1))U_\gamma(s_1,s_2)\tilde{\sigma}^{a_2}(\gamma(s_2))\cdots].$$

These observables are multitensors that transform as vector densities of weight one in each points $\gamma(s_i)$ and related index a_i, and as scalars in all the other points where γ is.

It may be then shown that the T_ϵ^n's form a closed Poisson algebra, and that this algebra is related to the $\tilde{\mathcal{T}}$ algebra by

$$\{T_\epsilon^n,T_\epsilon^m\}^\sim=\lim_{\hbar\rightarrow0}(i\hbar)^{-1}[\tilde{T}_\epsilon^n,\tilde{T}_\epsilon^m].$$

Thus, the space of loop functionals $\mathcal{S}$, together with the $\tilde{\mathcal{T}}$ algebra, provides a representation of the kinematics of quantum general relativity.[10] Recall that a quantum theory corresponding to a given classical theory may be defined by a linear representation of a deformation of *any* complete Poisson algebra of observables.[8]

The constraints of general relativity, which define the dynamics, are represented in the loop representation in the following ways. The SU(2) gauge constraints of the Ashtekar formalism are represented trivially, since the algebra $\mathcal{T}$ consists of only the SU(2) gauge-invariant observables.

To define the diffeomorphism constraints we begin by noting that the natural action $(\phi\cdot\gamma)(s)\equiv\phi(\gamma(s))$ of the diffeomorphisms $\phi\in\mathrm{Diff}(\Sigma)$ on the loop space $\mathcal{L}_\Sigma$ (and on $\mathcal{M}_\Sigma$) induces a linear representation U of the group $\mathrm{Diff}(\Sigma)$ on the space $\mathcal{S}$ of the loop functionals. This is given by

$$U(\phi)\mathcal{A}[\{\eta\}]\equiv\mathcal{A}[\phi^{-1}\cdot\{\eta\}]. \tag{2}$$

If ϕ_t is a one-parameter group of diffeomorphisms on Σ generated by the vector field v on Σ, the generators $D(v)$ of U are given by

$$D(v)\mathcal{A}[\{\eta\}]\equiv dU(\phi_t)\mathcal{A}[\{\eta\}]/dt\,|_{t=0};$$

on the domain on which the $D(v)$ are defined (which is a dense subset of $\mathcal{S}$) they satisfy $[D(v),D(w)]\equiv D([v,w])$. The action of a diffeomorphism on an element of $\tilde{\mathcal{T}}$ is given by $\phi\cdot\tilde{T}\equiv U(\phi)\tilde{T}U(\phi)^{-1}$. By explicit computation it may be shown that $\phi\cdot\tilde{T}^0[\gamma]=\tilde{T}^0[\phi^{-1}\cdot\gamma]$ and that also the other $\tilde{T}^n_\epsilon$'s transform exactly as the correspondent classical observables T^n_ϵ. Therefore we have two results. First, the regularization built into the definition of the $\tilde{\mathcal{T}}$ algebra is covariant under the action of the spatial diffeomorphisms. Second, since the $D(v)$ have the correct commutation relations with the observables (as follows from the transformation properties of the $\tilde{T}^n$) and among themselves, we can[8] identify them with the diffeomorphism constraints.

Finally, we represent the Hamiltonian constraint through a sequence of regulated operators. In the Ashtekar formalism the Hamiltonian constraint has the form $\mathcal{C}(x)=\mathrm{Tr}[F_{ab}(x)\tilde{\sigma}^a(x)\tilde{\sigma}^b(x)]$, where $F_{ab}(x)$ is the curvature of Ashtekar's connection. The presence of the square of the field operators is a sign that the corresponding quantum operator requires a suitable regularization. We recall that the holonomy of a small loop γ_δ of coordinate radius δ in the coordinate $(\hat{a},\hat{b})$ plane may be expanded in δ as $1+\delta^2F_{\hat{a}\hat{b}}+O(\delta^3)$ (1 is the identity). Therefore the curvature may be defined by a suitable limit of T^0's as the loops shrinks down to a point. Similarly $\mathcal{C}(x)$ can be expressed in terms of the classical observables T^n as a limit of a sequence of observables $\mathcal{C}^\delta(x)$ given by a suitable linear combination of T^2's. The details of this are described in Ref. 3. In the loop representation we define the Hamiltonian constraint as the limit of the quantum operators $\tilde{\mathcal{C}}^\delta(x)$. In this way the theory naturally provides the regulated form of the operator.

Before presenting the solutions of the quantum constraint equations we discuss here the relation between the loop representation and the self-dual representation of quantum gravity,[5] which is defined in terms of holomorphic functionals of the connection $\Psi(A)$. This relation is expressed by a linear mapping from the conjugate self-dual representation to the loop representation. The gauge conjugate representation consists of linear maps Φ from elements of the self-dual representation $\Psi(A)$ to the complex numbers, denoted $\Phi[\Psi(\cdot)]$ (the Dirac bras's). In terms of the functionals

$$H(\{\gamma\},A)=\prod_{\gamma\in\{\gamma\}}\mathrm{Tr}P\exp\left\{\oint_\gamma A\right\},$$

we define a mapping $\mathcal{T}$ from the conjugate self-dual representation to the loop representation by

$$\mathcal{T}:\Phi\to\mathcal{A}[\{\gamma\}]\equiv\Phi[H(\{\gamma\},\cdot)].$$

This equation can be written in a more intuitive, but formal, way if we introduce an arbitrary measure $\mu(A)$ on the space of connections, and so express any conjugate element Φ as $\Phi(A)$ by $\Phi[\Psi(\cdot)]=\int d\mu(A)\Psi(A)\Phi(A)$. The transform $\mathcal{T}$ is then expressed as

$$\mathcal{A}[\{\gamma\}]=\int d\mu(A)H(\{\gamma\},A)\Phi(A). \tag{3}$$

Operators in the self-dual representation, denoted $\hat{O}$, are related to the corresponding operators in the loop representation, denoted $\tilde{O}$, by the relation $\tilde{O}\mathcal{T}=\mathcal{T}\hat{O}^\dagger$. This will be true if

$$\hat{O}H(\{\gamma\},A)=\tilde{O}H(\{\gamma\},A),$$

where $\hat{O}$ acts on the connection while $\tilde{O}$ acts on the loop. This equation may be explicitly checked for all the observables and the constraints.

The integral (3) may be explictely computed in the linearized theory, where there is a natural choice for $\mu(A)$, and by doing so a complete representation of the Fock space of linearized general relativity in terms of loop functionals may be obtained.[11]

We may now finally discuss how solutions to the constraint equations are obtained. By Eq. (2) we know that, if $\mathcal{A}[\{\eta\}]$ is to solve the diffeomorphism constraint equation $D(v)\mathcal{A}[\{\eta\}]=0$, it must be constant on the orbits of $\mathrm{Diff}(\Sigma)$ in $\mathcal{M}_\Sigma$. Thus, if $L(\{\eta\})$ is defined to be the equivalence class of the multiple loop $\{\eta\}$ under $\mathrm{Diff}(\Sigma)$ (which we will call the generalized link class of $\{\eta\}$) then the general solution to the diffeomorphism constraints is

$$\mathcal{A}[\{\eta\}]=\mathcal{A}[L(\{\eta\})].$$

For example, let $\mathcal{P}$ be any ordinary link invariant,[4] then a solution to the diffeomorphism constraints is $\mathcal{A}_\mathcal{P}[\{\eta\}]\equiv 0$ if $\{\eta\}$ contains any intersections and, otherwise, $\mathcal{A}_\mathcal{P}[\{\eta\}]\equiv\mathcal{P}[\{\eta\}]$.

The solutions to the regulated Hamiltonian constraint are defined by the condition that, for every x and every $\{\eta\}$,

$$\lim_{\delta\to 0}\tilde{\mathcal{C}}^\delta(x)\mathcal{A}[\{\eta\}]=0. \tag{4}$$

In Ref. 3 the following result is demonstrated. $\mathcal{A}[\{\eta\}]$ satisfies Eq. (4) provided that $\mathcal{A}[\{\eta\}]=0$ whenever $\{\eta\}$ involves intersections or nondifferentiable points. Thus, there are simultaneous solutions to all of the constraints. The $\mathcal{A}_\mathcal{P}$ described above are examples of them.

While the construction of the regulated Hamiltonian constraint and the study of its action are too complicated to describe here, we may give a heuristic argument for the form of its solutions based on the transform. The key idea is that the kernel of the transform, $H(\{\gamma\},A)$, is annihilated by the Hamiltonian constraint of the self-dual representation whenever the curves $\{\gamma\}$ are nonintersecting and differentiable.[6] Thus an $\mathcal{A}[\{\eta\}]$ with support only on differentiable non-self-intersecting multiple loops may be thought of as a superposition of states $\Psi(A)=H(\{\gamma\},A)$ that solve the self-dual Hamiltonian constraint equation. The same argument suggests that other solutions may be given by loop functionals that have support also on intersecting loops but satisfy the algebraic conditions involving the intersections described in Refs. 6 and 12.

We close with some comments. (1) It is likely that the class of physical states so far found are only a sector of physical space; whether or not these methods will lead to a construction of the entire physical space is an open issue. (2) Two crucial steps are missing for a definition of a complete theory of quantum gravity. The first is the construction of the physical inner product. For a closed universe this problem is related to the difficulties in the meaning of time and probability in quantum cosmology. (3) The other is the definition of the gauge-invariant observables. Although the invariants of knot theory may be used to construct a large class of operators on the physical state space, the physical meaning of these operators is unclear, as no explicit physical observables are known for classical general relativity for a compact universe in the absence of matter. (4) Without a solution for these two problems the physical interpretation of the physical loop states that we have found is an open problem. The extension of the present results to the asymptotically flat case, and the inclusion of matter may throw light on these issues.

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[a]Present address: Department of Physics, Syracuse University, Syracuse, NY 13210.

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[2]For a review see K. Kuchar, in *Quantum Gravity 2,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford Univ. Press, Oxford, 1982).

[3]These results are described in more detail by C. Rovelli and L. Smolin, "Loop Representation of Quantum General Relativity" (to be published).

[4]See, for instance, L. H. Kauffman, *On Knots* (Princeton Univ. Press, Princeton, NJ, 1987), and references therein.

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[7]C. Rovelli, invited contribution to the review of Ashtakar, Ref. 5, Chap. V.8.

[8]C. J. Isham has stressed that for a nonperturbative quantization of general relativity a noncanonical Poisson algebra may have to be used. See, for example, C. J. Isham, in *Relativity Groups and Topology II,* edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

[9]R. Penrose, in *Combinatorial Mathematics and Its Applications,* edited by D. J. A. Welch (Academic, London, 1971).

[10]The loop representation gives also a representation of the physical operator algebra for an SU(2) Yang-Mills theory, and with suitable modifications, of any Yang-Mills theory. Work is in progress in this direction. For the Abelian case see A. Ashtekar, to be published.

[11]A. Ashtekar, C. Rovelli, and L. Smolin, "Linearized Quantum Gravity in Loop Space" (to be published).

[12]V. Husain, "Intersecting Loop Solutions to the Hamiltonian Constraint of Quantum General Relativity" (to be published).