Existence of Long-Range Order in the *XY* Model

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The existence of long-range order (LRO) at low temperature in the isotropic XY model with spin $S \geqq 1$ on simple cubic lattice in three or more dimensions is shown. This completes the proof of the existence of LRO at low temperature in the isotropic XY model with arbitrary spin size in three or more dimensions, as the proof for $S=\frac{1}{2}$ was given earlier by Dyson, Lieb, and Simon. The existence of LRO in the ground state is shown for $S \geqq \frac{3}{2}$ in two dimensions.

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Although the existence of phase transitions in the Heisenberg ferromagnets or antiferromagnets (AF) in three dimensions is taken for granted by many physicists, the rigorous proof of the existence is far from a simple problem. The proof for the Heisenberg ferromagnet, which is most likely to have the long-range order (LRO) at low temperature, is not yet known, while the proof for the classical counterpart is known.[1] The existence of phase transitions in quantum spin systems with a continuous symmetry group was first shown rigorously by Dyson, Lieb, and Simon (DLS).[2] They showed the existence of LRO at low temperature in the spin-$\frac{1}{2}$ isotropic XY model and the AF Heisenberg model with spin $S \geqq 1$ in three or more dimensions.

Recently the existence of LRO in the ground state in two dimensions has been questioned in the isotropic XY[3] and the AF Heisenberg[4] models from the numerical results of finite systems. The existence of LRO in the ground state was proved for the AF Heisenberg model in two dimensions for $S \geqq \frac{3}{2}$ by Neves and Perez.[5] Their proof applies also for $S=1$.[6]

In this Letter I report the existence of LRO at low temperature in the isotropic XY model with $S \geqq 1$ in three or more dimensions. Also LRO is shown to exist in two dimensions for $S \geqq \frac{3}{2}$.

The model we consider is described by the Hamiltonian

$$H=-\sum_{\boldsymbol{\alpha} \in \Lambda} \sum_{m=1}^{\nu} \sum_{i=1}^{2} S_{\boldsymbol{\alpha}}^i S_{\boldsymbol{\alpha}+\boldsymbol{\delta}_m}^i, \qquad (1)$$

where Λ is a hypercube in the simple ν-dimensional cubic lattice:

$$\Lambda=\{\boldsymbol{\alpha} \mid 0 \leqq \alpha_1 \leqq L-1, \ldots, 0 \leqq \alpha_\nu \leqq L-1\}.$$

There is a spin $S_{\boldsymbol{\alpha}}$ with size S on each site $\boldsymbol{\alpha} \in \Lambda$ interacting with the spins at $\boldsymbol{\alpha} \pm \boldsymbol{\delta}_i$, $i=1, \ldots, \nu$. Here $\boldsymbol{\delta}_i$ is the unit vector whose ith component is 1 and I use the convention that if $\alpha_i=L-1$ then $(\boldsymbol{\alpha}+\boldsymbol{\delta}_i)_i=0$.

I define

$$g_{\mathbf{p}}^i=\langle S_{\mathbf{p}}^i S_{-\mathbf{p}}^i\rangle$$

with

$$S_{\mathbf{p}}^i=\frac{1}{L^{\nu/2}} \sum_{\boldsymbol{\alpha} \in \Lambda} e^{i \mathbf{p} \cdot \boldsymbol{\alpha}} S_{\boldsymbol{\alpha}}^i, \quad \mathbf{p} \in \Lambda^* \text{ (dual lattice)},$$

where $\langle A\rangle$ denotes the expectation value of an observable A at inverse temperature β:

$$\langle A\rangle=\frac{1}{Z} \operatorname{Tr}[e^{-\beta H} A], \quad Z=\operatorname{Tr} e^{-\beta H}.$$

A ferromagnetic LRO, m^i, is defined as

$$[m^i]^2=\lim_{L \to \infty}\left\langle\left[\frac{1}{L^\nu} \sum_{\boldsymbol{\alpha} \in \Lambda} S_{\boldsymbol{\alpha}}^i\right]^2\right\rangle=\lim_{L \to \infty} \frac{1}{L^\nu} g_{\mathbf{p}=0}^i. \qquad (2)$$

Existence of LRO, i.e., $m^i \neq 0$ is equivalent to

$$\lim_{L \to \infty} \frac{1}{L^\nu} \sum_{\mathbf{p} \neq 0} g_{\mathbf{p}}^i<\langle S_{\boldsymbol{\alpha}}^i S_{\boldsymbol{\alpha}}^i\rangle$$

because of the sum rule

$$\frac{1}{L^\nu} \sum_{\mathbf{p} \in \Lambda^*} g_{\mathbf{p}}^i=\langle S_{\boldsymbol{\alpha}}^i S_{\boldsymbol{\alpha}}^i\rangle.$$

With help of the Falk-Bruch inequality,[7] a sufficient condition for (2) was derived and stated as Theorem 5.1 of DLS.[2] The theorem is given as follows:

Suppose the model (1) satisfies the following conditions:

(i) "Gaussian domination,"

$$(S_{\mathbf{p}}^i, S_{-\mathbf{p}}^i) \leqq B_{\mathbf{p}}^i / \beta, \quad \mathbf{p} \neq 0 \text{ for all } \beta, \qquad (3)$$

where

$$(A, B)=Z^{-1} \int_0^1 \operatorname{Tr}(e^{-x \beta H} A e^{-(1-x) \beta H} B) d x$$

is the Duhamel two-point function.

(ii) Existence of an upper bound for the expectation of the double commutator,

$$\langle[S_{\mathbf{p}}^i,[H, S_{-\mathbf{p}}^i]]\rangle \leqq C_{\mathbf{p}}^i. \qquad (4)$$

(iii) Existence of a lower bound for the usual two-point function

$$\langle S_{\boldsymbol{\alpha}}^i S_{\boldsymbol{\alpha}}^i\rangle \geqq D^i(\beta). \qquad (5)$$

Then there is LRO at any β such that

$$D^i(\beta) > \frac{1}{2(2\pi)^\nu}\int_{|p_i|<\pi}(B^i_{\mathbf{p}}C^i_{\mathbf{p}})^{1/2}\coth[\tfrac{1}{2}\beta(C^i_{\mathbf{p}}/B^i_{\mathbf{p}})^{1/2}]d^\nu\mathbf{p}. \tag{6}$$

One easily sees from (6) that $B^i_{\mathbf{p}}$ and $C^i_{\mathbf{p}}$ must satisfy the condition

$$\frac{1}{(2\pi)^\nu}\int_{|p_i|<\pi}(B^i_{\mathbf{p}}C^i_{\mathbf{p}})^{1/2}d^\nu p < \infty$$

as $\coth x > 1$ $(x>0)$. It must be noted that $m^i \neq 0$ does not immediately imply existence of a spontaneous magnetization since the total magnetization $\sum_{\alpha\in\Lambda}S^i_\alpha$ ($i=1$ or 2) does not commute with the Hamiltonian (1).[2] Even in such a case, it was shown by DLS that existence of a phase transition can be deduced from inequality (6) by showing the absence of clustering.

I prove that (6) holes for $i=1$ at $\beta>\beta_c$ for some $\beta_c<\infty$ for $S\geqq 1$ and $\nu\geqq 3$. I fully utilize below the symmetry properties, i.e., $\langle (S^i_\alpha)^2\rangle = \langle (S^j_\beta)^2\rangle$ and

$$\langle S^i_\alpha S^i_{\alpha+\delta_m}\rangle = \langle S^j_\alpha S_{\alpha+\delta_{m'}}\rangle$$

for any $1\leqq i,\ j\leqq 2$, $\boldsymbol{\alpha}$, β, m, and m'. These relations are easily shown to hold the use of obvious unitary transformations. The condition (i) is satisfied with $B^1_{\mathbf{p}} = (2E_{\mathbf{p}})^{-1}$ where

$$E_{\mathbf{p}} = \sum_{m=1}^{\nu}[1-\cos(\boldsymbol{\delta}_m\cdot\mathbf{p})], \tag{7}$$

as was shown by DLS. I take $C^1_{\mathbf{p}}$ as the expectation itself:

$$C^1_{\mathbf{p}} = \langle [S^1_{\mathbf{p}},[H,S^1_{-\mathbf{p}}]]\rangle = 2L^{-\nu}\sum_{\alpha}\sum_{m=1}^{\nu}\{\langle S^2_\alpha S^2_{\alpha+\delta_m}\rangle - \cos(\mathbf{p}\cdot\boldsymbol{\delta}_m)\langle S^3_\alpha S^3_{\alpha+\delta_m}\rangle\}. \tag{8}$$

The right-hand side of the inequality (6) is evaluated as

$$I^1_\nu \equiv \frac{1}{(2\pi)^\nu}\int d^\nu p\,\tfrac{1}{2}(B^1_{\mathbf{p}}C^1_{\mathbf{p}})^{1/2}\coth[\tfrac{1}{2}\beta(C^1_{\mathbf{p}}/B^1_{\mathbf{p}})^{1/2}] < \frac{1}{(2\pi)^\nu}\int d^\nu p[\tfrac{1}{2}(B^1_{\mathbf{p}}C^1_{\mathbf{p}})^{1/2}+\beta^{-1}B_{\mathbf{p}}]. \tag{9}$$

Here I have used $\coth x < 1 + x^{-1}$ for $x>0$.[2] For $\nu\geqq 3$,

$$G_\nu(0) \equiv \frac{1}{(2\pi)^\nu}\int\frac{d^\nu p}{E_{\mathbf{p}}} < \infty. \tag{10}$$

Therefore it follows that

$$I^1_\nu < \frac{1}{2}\left(\frac{G_\nu(0)}{2}\frac{1}{(2\pi)^\nu}\int C^1_{\mathbf{p}}d^\nu p\right)^{1/2} + \frac{1}{2\beta}G_\nu(0) = (\tfrac{1}{4}\nu G_\nu(0)\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle)^{1/2} + \frac{1}{2\beta}G_\nu(0). \tag{11}$$

Here the Schwarz inequality

$$\left[\frac{1}{(2\pi)^\nu}\int d^\nu p(B^1_{\mathbf{p}}C^1_{\mathbf{p}})^{1/2}\right]^2 \leqq \left(\frac{1}{(2\pi)^\nu}\int d^\nu p B^1_{\mathbf{p}}\right)\left(\frac{1}{(2\pi)^\nu}\int d^\nu p C^1_{\mathbf{p}}\right)$$

has been employed. Also the second term of (8) vanishes after $\mathbf{p}$ summation and the symmetry relation $\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle = \langle S^2_\alpha S^2_{\alpha+\delta_{m'}}\rangle$ has been used.

From the symmetries, $\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle$ is related to the expectation of the energy as

$$\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle = -(2\nu)^{-1}L^{-\nu}\langle H\rangle,$$

and therefore

$$\lim_{\beta\to\infty}\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle = -(2\nu)^{-1}L^{-\nu}\langle H\rangle_0,$$

where $\langle H\rangle_0$ is the ground-state energy of the system. Since the wave function with all the spins directed in the same direction gives the upper bound of $\langle H\rangle_0$ as $-\nu L^\nu S^2$, we have

$$\lim_{\beta\to\infty}\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle \geqq \tfrac{1}{2}S^2. \tag{12}$$

As $\nu G_\nu(0)$ is a decreasing function[2] of ν and $\frac{3}{4}G_3(0) = 0.379 < \frac{1}{2}S^2$ for $S\geqq 1$, it can be deduced from (11) and (12) that there exists an inverse temperature $\beta_c<\infty$ such that

$$I^1_\nu < \langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle$$

holds at $\beta>\beta_c$. Taking account of the Schwarz inequality

$$[\langle (S^1_\alpha)^2\rangle]^2 = \langle (S^1_\alpha)^2\rangle\langle (S^1_{\alpha+\delta_m})^2\rangle \geqq (\langle S^1_\alpha S^1_{\alpha+\delta_m}\rangle)^2, \tag{13}$$

one concludes that (6) holds at $\beta>\beta_c$.

It is well known that no LRO exists in two dimensions at finite temperature.[8] It is sufficient for LRO to exist in the ground state that the $T=0$ version of (6) is satisfied,[5] i.e.,

$$D^1(\beta=\infty) > \frac{1}{2(2\pi)^2}\int d^2p(B^1_{\mathbf{p}}C^1_{\mathbf{p}})^{1/2}. \tag{14}$$

One has $|\langle S_a^3 S_{a+\delta_m}^3\rangle| \leqq \langle S_a^1 S_{a+\delta_m}^1\rangle$ applying $C_{\mathbf{p}}^1 \geqq 0$ to (8) for $\mathbf{p}=0$ and (π,π). By use of the upper bound

$$C_{\mathbf{p}}^1 \leqq 2\langle S_a^1 S_{a+\delta_m}^1\rangle \sum_{m=1}^{2} \{1+|\cos(\mathbf{p}\cdot\boldsymbol{\delta}_m)|\},$$

a sufficient condition for (14) is written as

$$\sqrt{2}S > \frac{1}{(2\pi)^2}\int d^2p \left[\frac{2+|\cos p_1|+|\cos p_2|}{2-\cos p_1-\cos p_2}\right]^{1/2}.$$

As the right-hand side is evaluated numerically as 1.67, LRO exists for $S \geqq \frac{3}{2}$.

I have shown above that LRO exists at sufficiently low temperature for $S \geqq 1$ and $\nu \geqq 3$, which implies the existence of a phase transition at finite temperature.[2] Also the existence of LRO has been proved in the ground state in two dimensions for $S \geqq \frac{3}{2}$. In the proof I have assumed that the thermal averages are continuous functions of β and approach the value in the ground state when $\beta \rightarrow \infty$. If the ground state is not singlet the proof applies for equally weighted averages over the ground states for $\nu=2$.

In the above proof the condition (6) has been related to the upper bound of $\langle H\rangle_0$ by use of the Schwarz inequality (13) rather than the lower bound employed in earlier works.[2,5] The same argument gives LRO for $S \geqq \frac{3}{2}$ in three and more dimensions and for $S \geqq 2$ in two dimension for the AF Heisenberg model. These cases are included in already known results.

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