Lyapunov Spectrum of a Model of Two-Dimensional Turbulence

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A scalar model of two-dimensional Navier-Stokes turbulence first proposed by Gledzer is shown to realize the power law $E(k) \sim k^{-3}$ in its *chaotic* state, which is found to obey the same scaling law as that of the enstrophy-cascade theory. All the Lyapunov exponents are calculated for several values of viscosity, and they are found to have a scaling property in the interior of the attractor. The calculated distribution function of the Lyapunov exponents appear to have a singularity at null Lyapunov exponent.

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Properties of fully developed turbulence still lack an appropriate interpretation in the strange-attractor theory in spite of recent advances in the theory of dynamical systems. In particular, the scaling property in the inertial range has not yet been understood as any characteristic of such a strange attractor, mostly because numerical techniques developed so far are not sufficiently powerful for high-dimensional attractors of partial differential equations because of insufficient ability of present computers. At the present stage, therefore, one of the possible strategies is to investigate in detail a chaotic dynamical system which has a scaling property similar to that in real fluid turbulence but is tractable in size. In this direction Grappin and co-workers[1,2] investigated a model equation of MHD turbulence, in which both the Kaplan-Yorke dimension of the attractor and the time-averaged kinetic and magnetic spectra are compatible with Kolmogorov's scaling law for fluid turbulence.

In this Letter we focus our attention on what type of strange attractor can be connected with the scaling law of turbulence. We investigate a model equation of two-dimensional (2D) turbulence of ordinary fluid which was first proposed by Gledzer.[3] The model is constructed in wave-number space, which is discretized as $K_n = k_0 q^n$ $(q > 1,\ 1 \leqq n \leqq N)$. The velocity is expressed by a set of real collective variables $\{u_n\}$, where u_n stands for the velocity components whose wave numbers k lie between k_n and k_{n+1}, $k_n < |k| < k_{n+1}$. The energy E and the enstrophy Q are therefore defined as $E = \sum_n u_n^2/2$ and $Q = \sum_n k_n^2 u_n^2/2$, and the energy spectrum $E(k_n)$ is $E(k_n) = u_n^2/2k_n$. Each evolution equation for u_n is assumed to be quadratically nonlinear and connected with the two preceding and the two succeeding equations. Moreover, the conservation of phase volume $\sum_n \partial \dot{u}_n/\partial u_n = 0$ is also assumed to hold in the inviscid unforced case, where the dot denotes the time derivative. These conditions yield the following evolution equation of $\{u_n : 1 \leqq n \leqq N\}$:

$$(d/dt + \nu k_n^2 + \nu' k_n^{-1}\theta_n) u_n = c_n^{(1)} u_{n+1} u_{n+2} + c_n^{(2)} u_{n-1} u_{n+1} + c_n^{(3)} u_{n-1} u_{n-2} + f^{(1)}\delta_{n,10} + f^{(2)}\delta_{n,11}, \tag{1}$$

$$c_n^{(1)} = c_n, \quad c_n^{(2)} = \frac{-c_{n-1}(k_{n-1}^2 - k_{n+1}^2)}{k_n^2 - k_{n+1}^2}, \quad c_n^{(3)} = \frac{c_{n-2}(k_{n-2}^2 - k_{n-1}^2)}{k_{n-1}^2 - k_n^2},$$
$$c_1^{(2)} = c_1^{(3)} = c_2^{(3)} = c_{N-1}^{(1)} = c_N^{(1)} = c_N^{(2)} = 0, \quad c_n = c_0 q^n, \tag{2}$$

where $c_n^{(1)}$, $c_n^{(2)}$, $c_n^{(3)}$ are determined so that both the energy and the enstrophy are conserved, $f^{(1)}$ and $f^{(2)}$ are time-independent forcing terms, ν is the kinetic viscosity, ν' is the dissipation coefficient with θ_n being unity for $n = 1$–9 and zero otherwise, δ is Kronecker's δ, and t is the time. The last term on the left-hand side of (1) is introduced to prevent the energy from increasing indefinitely by inverse cascade.[4]

We investigate the unsteady solutions of (1) numerically. Time marching is performed by the fourth-order Runge-Kutta method. In the following, we show the numerical results for $k_0 = c_0 = 2^{-10}$, $q = 2$, $f^{(1)} = f^{(2)} = 0.002$, $\nu' = 9\times 10^{-7}$, and $(\nu, N) = (10^{-6}, 22)$, $(10^{-7}, 24)$, $(10^{-8}, 26)$, $(10^{-9}, 27)$, $(10^{-10}, 29)$, $(10^{-12}, 32)$, $(10^{-14}, 36)$, and $(10^{-15}, 37)$. The numerical calculation was carried out in double-precision arithmetic on the vectorial computer VP-200 at Kyoto University.

We started the numerical integration with an arbitrarily chosen initial condition. After the initial transient period, an unsteady but apparently stationary state is realized, with the energy and enstrophy fluctuating respectively between 1 and 3 and between 0.5 and 2.5. The quantities we discuss below are evaluated in this apparently stationary state.

A scaling property for 2D turbulence was proposed by Batchelor and Kraichnan and Leith (BKL scaling)[5]; the enstrophy dissipation wave number k_d and the energy spectrum $E(k)$ in the inertial range are expressed as

$$k_d = \eta^{1/6}\nu^{-1/2} E(k) = \eta^{1/6}\nu^{3/2} E_e(k/k_d),$$

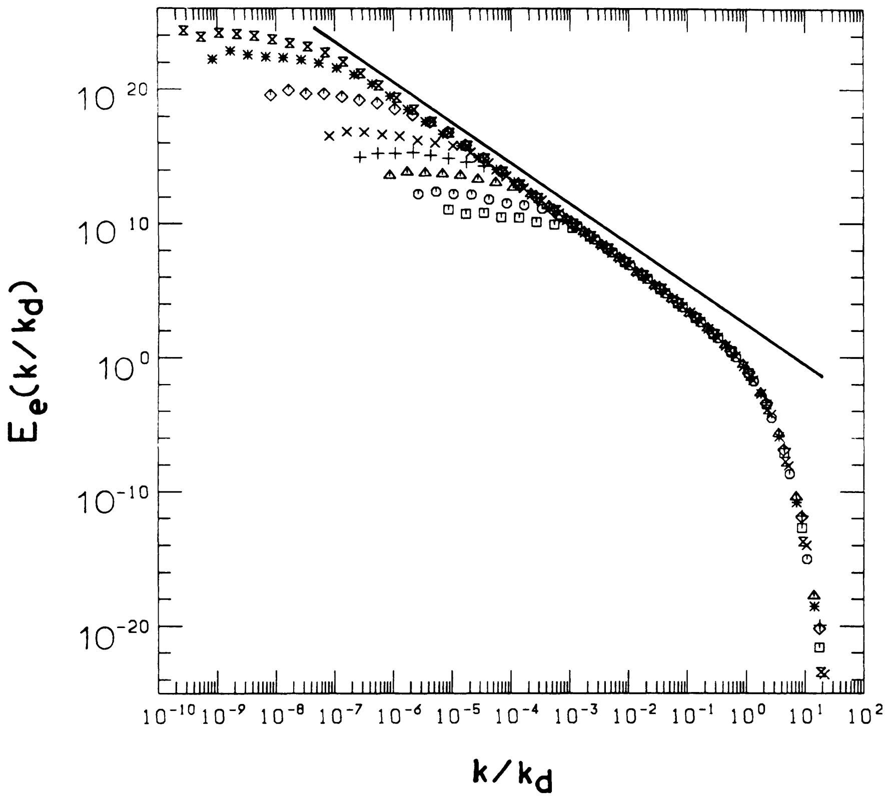

FIG. 1. Normalized energy spectrum function $E_e(k/k_d)$ for $\nu = 10^{-6}$ (squares), 10^{-7} (circles), 10^{-8} (triangles), 10^{-9} (plusses), 10^{-10} (crosses), 10^{-12} (lozenges), 10^{-14} (asterisks), and 10^{-15} (double triangles). The straight line shows slope -3.

where η, ν, and k denote the enstrophy dissipation rate, the kinematic viscosity, and the wave number, respectively, and E_e is a nondimensional function. According to the BKL scaling law the energy spectrum takes a power form $E(k) \sim \eta^{2/3} k^{-3}$ in the inertial subrange. We obtain the energy spectrum $E(k)$ by averaging the instantaneous energy spectrum over the time interval $0 \leqq t \leqq 12000$. The mean enstrophy dissipation rate η is then evaluated from this spectrum as $\eta = 2\nu \sum_n k_n^4 E(k_n)$. We see in Fig. 1 that the energy spectra normalized following the BKL scaling law for several values of viscosity agree fairly well and that the enstrophy inertial subrange with k^{-3} spectrum is remarkably realized, which shows that the BKL scaling law for 2D Navier-Stokes turbulence is also embodied in the model (1).

We calculated all the Lyapunov exponents λ_j and the Lyapunov vectors $v_n^{(j)}$ $(1 \leqq n, j \leqq N)$ in the stationary state making use of the linearized version of (1) and the Gram-Schmidt orthogonalization method, where we took $(\sum_n v_n^{(j)2})^{1/2}$ as the norm of $v_n^{(j)}$ and the Lyapunov exponents are ordered as $\lambda_j \geqq \lambda_{j+1}$. The numerical integration was performed until a plausible convergence was obtained, and it was checked that $\sum_j \lambda_j = -\nu \sum_j k_j^2 - \nu' \sum_{j=1}^{9} k_j^{-1}$ holds with an error less than 0.01%. The values of the Lyapunov exponents thus obtained vary with the viscosity, but in every case we calculated, some of the Lyapunov exponents are positive, which indicates that the velocity u_n moves *chaotically* on a strange attractor. The spectrum of the first Lyapunov vector exhibits a k^{-s} spectrum $(s \sim 2)$ in the inertial subrange in

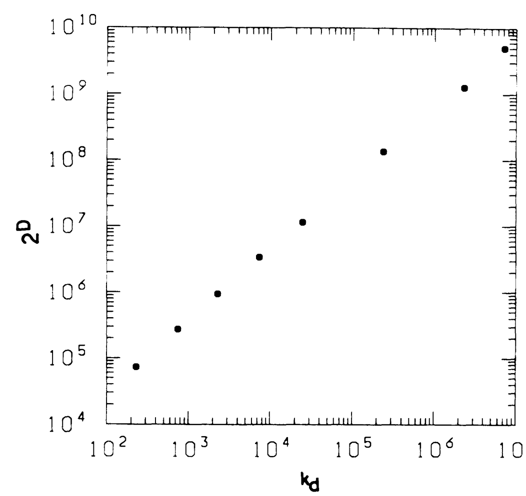

FIG. 2. Kaplan-Yorke dimension D vs the enstrophy dissipation wave number k_d.

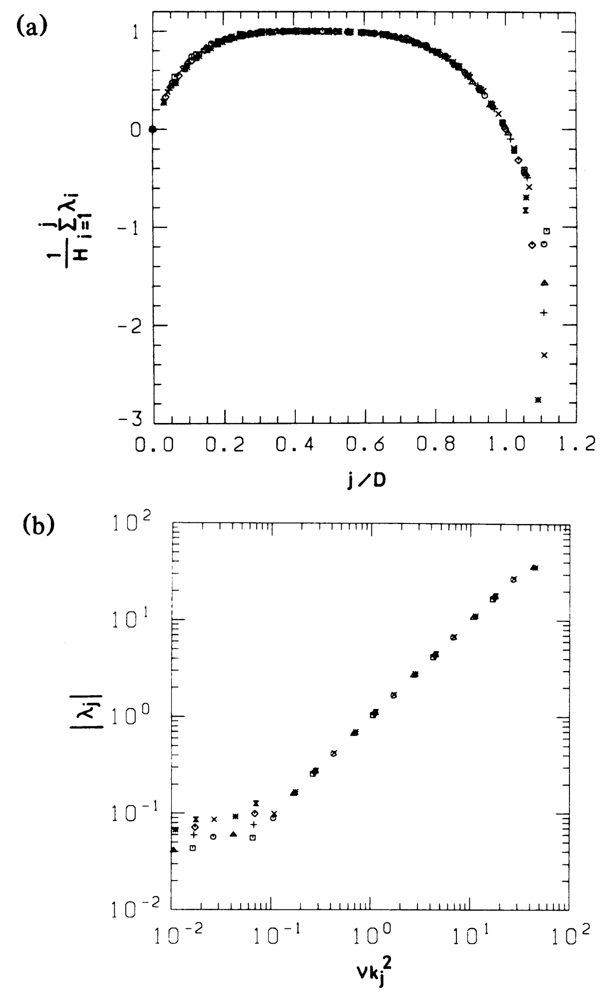

FIG. 3. (a) Normalized Lyapunov exponents. (b) Lyapunov exponents λ_j for large j vs the viscous dissipation rates at the wave number k_j.

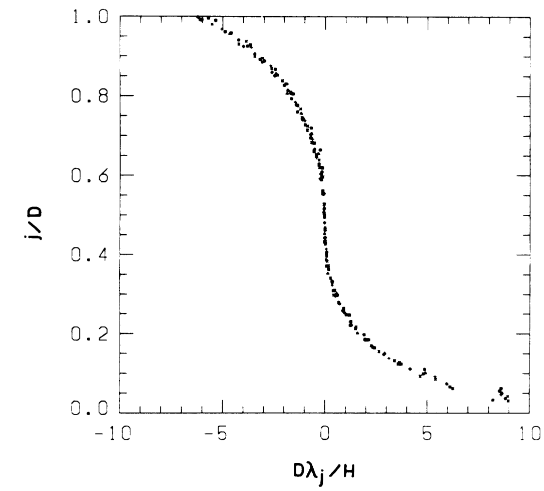

FIG. 4. Distribution of the Lyapunov exponents.

agreement with the result of a full simulation of the 2D Navier-Stokes equation.[6]

The Kaplan-Yorke dimension D of the attractor is $D=p+\sum_{j=1}^{p}\lambda_j/|\lambda_{p+1}|$, $p=\max\{m\,|\,\sum_{j=1}^{m}\lambda_j \geqq 0\}$.[7] We see in Fig. 2 that the dissipation wave number is proportional to 2^D which means that the BKL scaling law of the enstrophy dissipation wave number in real 2D turbulence also holds in the model (1) since $k_n=2^{n-10}$. Another quantity characterizing a chaotic attractor is the Kolmogorov entropy $H=\sum_{j=1}^{q}\lambda_j$ ($\lambda_q>0$, $\lambda_{q+1}\leqq 0$). The Kolmogorov entropy and the largest Lyapunov exponent appear to behave as $\log(1/\nu)$.

We plot in Fig. 3(a) $\sum_{i=1}^{j}\lambda_i$ normalized by the Kolmogorov entropy H, taking j/D as ordinate. It is seen that the graphs for several values of the viscosity agree well for the ordinates j/D less than 1, while they scatter for j/D larger than 1. This means that the Lyapunov exponents due to the interior of the attractor ($j/D<1$) have a definite scaling law. On the other hand, the Lyapunov exponents for $j/D \gg 1$ do not have the above scaling property but are strongly related to the viscous damping rate νk_j^2 as shown in Fig. 3(b), where $|\lambda_j|$ is plotted as a function of νk_j^2. The coincidence of the Lyapunov exponent with the linear damping rate has been attributed as a characteristic of the exterior of the attractor in the case of a time-delayed differential equation describing an optical phenomenon in laser systems.[8]

It is interesting that the scaling law of the Lyapunov exponents with respect to H and D occurs in the wave-number range lower than k_d, different from the BKL scaling law. The scaling law of the Lyapunov exponents in the interior of the attractor leads us to that of the distribution function of the Lyapunov exponents. In Fig. 4 we can see a nondimensional function f such that $j/D=f(D\lambda_j/H)$, where j is the index of the Lyapunov exponent and is also the number of Lyapunov exponents between λ_j and λ_1, $\mathrm{Card}\{\lambda_i\,|\,\lambda_j \leqq \lambda_i \leqq \lambda_1\}$. The distribution function which is proportional to $-df(\lambda)/d\lambda$ appears to diverge at null Lyapunov exponent. A similar divergence has been addressed by Ruelle for 3D Navier-Stokes turbulence,[8] while care should be taken of the possibility that there might be several mechanisms which lead to such a divergence. The divergence of the distribution function was also found by Ikeda and Matsumoto in the time-delayed differential equation,[9] while in the case of the Kuramoto-Sivashinsky equation such divergence was not observed.[10] Further analysis on this singularity is now under way and will be reported elsewhere together with a detailed study of this model and its 3D counterpart.

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