Better Way to Measure f_π in the Linear σ Model

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It is argued that the zero-momentum two-point function in a finite box can be used to get f_π in the linear σ model. The method is based on a soft-pion finite-size theorem which is conjectured to hold if the boundary conditions are periodic. The present procedure is expected to be much more efficient than a previous proposal.

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Several years ago it was conjectured on the basis of the triviality of φ^4 that, within the standard model, the Higgs-boson mass cannot exceed an approximately calculable upper bound.[1] The method proposed for our obtaining a lattice estimate of the bound to leading order in the weak gauge coupling required a set of Monte Carlo measurements on a latticized Gell-Mann–Levi linear σ model. The Lagrangean density, generalized to O(N), is given in Minkowski space by

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\boldsymbol{\phi}\partial^\mu\boldsymbol{\phi}-\tfrac{1}{2}m_0^2\boldsymbol{\phi}^2-(4N)^{-1}g_0^2(\boldsymbol{\phi}^2)^2. \qquad (1)$$

Getting the bound would also imply that (1) becomes trivial when the UV cutoff is removed.

The proposal was inspired by Freedman, Smolensky, and Weingarten,[2] who showed triviality for the $N=1$ case in the symmetric phase. It was shown there that, if the UV cutoff is taken to infinity while the bare mass is adjusted so that the physical mass is kept fixed, then, for any tuning of the wave-function renormalization and of the bare coupling which gives finite, nontrivial Green's functions at finite distances in the inverse physical mass, the physical four-point coupling is forced to vanish at infinite cutoff.

In the broken sector there is no finite correlation length. If $\langle\phi^i\rangle=\delta^{i1}\langle\sigma\rangle$, $\boldsymbol{\phi}=(\sigma,\boldsymbol{\pi})$, the pions are massless and the σ particle is unstable. The most natural choice for the quantity to be kept fixed instead of the mass of the symmetric phase is the physical pion "decay constant" f_π. It is defined as follows: The currents $J_\mu^\alpha(x)$ which correspond to rotations in the 1-α planes, $\alpha=2,\ldots,N$, are normalized by the requirement that the corresponding charges close the same algebra as the matrices $T_{ij}^{[pq]}=\delta_{pi}\delta_{qj}-\delta_{pj}\delta_{qi}$. Let $|\pi^\beta(k)\rangle$ be normalized, on-shell, pion states of four-momentum k; then

$$\langle 0|J_\mu^\alpha(x)|\pi^\beta(k)\rangle=if_\pi k_\mu\delta^{\alpha\beta}\exp(-ik\cdot x). \qquad (2)$$

Equation (2) is appropriate even if we still keep an O(3,1)-invariant UV cutoff in the model. f_π is equal to the vacuum expectation value, v, of the renormalized σ field, up to a finite, nonvanishing multiplicative factor. Triviality now means that, when we take the UV cutoff to infinity while keeping the physical f_π fixed, we shall always, independently of any other adjustments we make (as long as the Green's functions stay finite, nontrivial functions of distances measured in f_π^{-1}), end up, at infinite cutoff, with one additional stable, massless particle corresponding to the σ field. To see that this implies triviality, I introduce temporarily a small explicit symmetry-breaking term which gives the pions a finite mass m_π^2. I am coming from the broken phase with $v\neq 0$ and there one has the exact relation[3] $\mu_\sigma^2-\mu_\pi^2=2\lambda^2v^2$. In this relation λ is an independently defined four-point coupling where, for $N=4$, $\lambda_0^2=g_0^2/4$. μ_σ^2 (μ_π^2 is defined similarly) is a parameter related to the full σ two-point function Δ_σ; in momentum space we have

$$\Delta_\sigma(p^2)=\int_{(2m_\pi)^2}^{\infty}\frac{dm^2}{p^2-m^2+i0}\rho_\sigma(m^2);\quad \mu_\sigma^{-2}=\int_{(2m_\pi)^2}^{\infty}\frac{dm^2}{m^2}\rho_\sigma(m^2). \qquad (3)$$

Triviality follows upon the removal of the explicit symmetry breaking whose sole role was to extract the two-pion-cut contribution from μ_σ^{-2}.

On the lattice Eq. (2) cannot be directly used to measure f_π. f_π could be obtained from the power decay of current-current correlations but it is better to avoid dealing with composite operators if possible. A way was suggested in Ref. 1; it was based on the following:

$$\langle 0|\pi^\alpha(x)|\pi^\beta(k)\rangle\equiv z\delta^{\alpha\beta}\exp(-ik\cdot x)\Longrightarrow\langle\sigma\rangle=f_\pi z,\quad \Delta_\pi^{-1}(p^2)\underset{p^2\to 0}{\sim}z^{-2}p^2. \qquad (4)$$

While now only simple expectation values are involved, two measurements are needed to get f_π: one to obtain $\langle\sigma\rangle$ and the other, z. Both measurements are potentially severely sensitive to finite-size effects and one may worry that, in practice, prohibitively large volumes would become necessary.

The purpose of this note is to propose a new way of measuring f_π. While it is still true that only the simplest correlations are needed, only one type of measurement is necessary and the finite-size effects, rather than posing a problem, are judiciously exploited. The method works for periodic boundary conditions in all four directions.

The basic idea is very simple. In a finite box there is no spontaneous symmetry breaking; only a measurement of $G_L(x)=\langle\phi(x)\phi(0)\rangle$ really makes sense. Spontaneous symmetry breaking is a property of $G_L(x)$ in the limit $L\to\infty$:

$$\lim_{L\to\infty} G_L(x) \underset{x\to\infty}{\sim} a+\frac{b}{x^2}+c\frac{(\ln x)^d}{x^4}. \tag{5}$$

Clearly, $a=(f_\pi z)^2$ while $b=3(N-1)z^2/8\pi^2$. Equation (5) implies

$$L^{-4}\sum_x G_L(x) \underset{L\to\infty}{\sim} a+\frac{b'}{L^2}+O\left(\frac{(\ln L)^{d'}}{L^4}\right). \tag{6}$$

Unfortunately b' is not related in a simple way to b because the limit $L\to\infty$ and the asymptotic expansion cannot be interchanged.

Nevertheless b' does contain a factor of z^2 and therefore it is natural to define a physical quantity c by $b'=z^2c$. c measures the leading dependence on the boundaries. This dependence must come from the pions in the box of wavelength of the order $1/L$. Soft pions interact weakly, and to leading order, all their scatterings can be parametrized by a single dimensional parameter,[4] f_π. Thus it is plausible to expect c to depend on the couplings of the model only through f_π. But the physical parameter c is dimensionless and, hence, cannot depend on f_π either. We are thus led to the expectation that c depends only on group theory (N), and kinematically, on the box shape (if all the sides go to infinity in fixed ratios of order 1; I shall deal only with the symmetrical case). So it should be possible to calculate c once and for all and to get f_π by observation of the leading size dependence in

$$L^{-4}\sum_x G_L(x) \underset{L\to\infty}{\sim} z^2\left(f_\pi^2+\frac{c}{L^2}\right)+O\left(\frac{(\ln L)^{d'}}{L^4}\right). \tag{7}$$

In order to test the validity of the arguments about the universality of c and to obtain an estimate of its value, I turn to the $1/N$ expansion. If we latticize and Euclideanize (1), the action becomes

$$S=\tfrac{1}{2}\sum_x\phi(x)\left[\sum_\mu\Delta_\mu\bar{\Delta}_\mu-m_0^2\right]\phi(x)-(4N)^{-1}g_0^2\sum_x[\phi^2(x)]^2. \tag{8}$$

Δ_μ ($\bar{\Delta}_\mu$) is the forward (backward) finite difference in the μ direction. Sums over sites x or over momenta r,s are always taken over integer-valued four-vectors with each component ranging from 0 to $L-1$. Now

$$G_L(x)=NL^{-4}\sum_s\exp\left(\frac{2\pi i}{L}s\cdot x\right)\Delta(s). \tag{9}$$

An effective "pion mass" m_L^2 is defined in terms of the couplings by

$$m_L^2=m_0^2+g_0^2L^{-4}\sum_r D(r),$$
$$D(r)=\left[4\sum_\mu\sin^2[(\pi/L)r_\mu]+m_L^2\right]^{-1}. \tag{10}$$

For sufficiently negative m_0^2, m_L^2 vanishes as $1/L^4$ for $L\to\infty$:

$$1/m_L^2L^4=A-\alpha/L^2+O(\ln L/L^4). \tag{11}$$

A depends on the couplings but α does not; $\alpha=-0.1405\ldots$. The evaluation of α (see below) depends only on the infrared behavior of $D(r)$ and is therefore UV-cutoff independent. At $N=\infty$, $\Delta(s)=D(s)$, and in view of (10), (7) holds with $c=-N\alpha$. Since $z=1$ at $N=\infty$, I feel that (7) is insufficiently tested.

To order $1/N$ we obtain after some algebra

$$\Delta^{-1}(0)=m_L^2+\frac{2}{N}\left[E(0)-\frac{1}{B(0)+g_0^{-2}}\frac{1}{L^4}\sum_r D^2(r)E(r)\right],$$
$$B(s)=L^{-4}\sum_r D(s-r)D(r),\quad E(s)=\sum_r[B(r)+g_0^{-2}]^{-1}D(s-r). \tag{12}$$

To the needed order in $1/L$ (12) simplifies:

$$L^{-4}\Delta(0)=\frac{1}{m_L^2L^4}-\frac{2}{N}\left\{L^{-4}\sum_{r\neq0}D^2(r)[E(0)-E(r)]+g_0^{-2}E(0)\right\}$$
$$=A-\frac{\alpha}{L^2}\left(1+\frac{2\hat{\mathcal{E}}-1}{N}\right)-\frac{2}{N}\int_p\mathcal{D}^2(p)\left[\mathcal{E}(0)-\mathcal{E}(p)-\frac{A}{\mathcal{B}(p)+g_0^{-2}}\right], \tag{13}$$

where $\int_p \equiv \int d^4p/(2\pi)^4$ and

$$\mathcal{D}(q)=1/\left[4\sum_\mu \sin^2(q_\mu/2)\right],\quad \mathcal{B}(q)=2A\mathcal{D}(q)+\int_p \mathcal{D}(q-p)\mathcal{D}(p),$$

$$\mathcal{E}(q)=\int_p \mathcal{D}(q-p)\frac{1}{\mathcal{B}(p)+g_0^{-2}} \underset{q^2\to 0}{\sim} \mathcal{E}(0)-q^2\hat{\mathcal{E}}. \tag{14}$$

For $k=2\pi r/L$ kept fixed and with $k^2\neq 0$ we find at $L=\infty$

$$\Delta^{-1}(r)\to\tilde{\Delta}^{-1}(k)=[1+(1-2\hat{\mathcal{E}})/N]k^2+O(k^4\ln k^2). \tag{15}$$

$\tilde{\Delta}^{-1}(k)$, being the inverse of the sum of the pion and σ propagators, contains a subleading logarithmic term representing the two-pion decay mode of σ.[5]

Comparing with (4) I confirm (7) to subleading order in $1/N$ with

$$c=-(N-1)\alpha,\quad \alpha=-\frac{1}{2\pi}\left[1-\sum_n{}'\frac{1}{\pi n^2}e^{-\pi n^2}\right]. \tag{16}$$

The unbounded sum over n runs over all integer four-vectors except $n=0$; it converges rapidly. α has arisen from

$$L^{-4}\sum_{s\neq 0}f(s)-\int_q \tilde{f}(q) \underset{L\to\infty}{\sim} \frac{\alpha}{L^2}, \tag{17}$$

where $f(s)=\tilde{f}(2\pi s/L)$ and $\tilde{f}^{-1}(q)\approx q^2$ at small q. The $1/N$ correction had the least effect it possibly could have: It restored the proper counting of pion states. It should be quite clear by now that (16) is very likely exact for any N. For $N=4$ we obtain

$$\frac{1}{L^4}\sum_x G_L(x) \underset{L\to\infty}{\sim} z^2(f_\pi^2+0.42/L^2)+O((\log L)^{d'}/L^4). \tag{18}$$

For L sufficiently large that the correction is unimportant we should be able to measure f_π easily if $f_\pi L\approx 1$. This makes f_π a relatively large number in inverse lattice spacings or, in other words, it is practical to go to relatively high UV cutoffs, which is what we would like to be able to do.

I now present an outline of a proof that

$$\langle M^2\rangle \equiv L^{-4}\sum_x G_L(x)=z^2[f^2-(N-1)\alpha/L^2]+\text{lower order}$$

to any order in perturbation theory. By consideration of the infinite-volume, leading infrared behavior of the pion and σ propagators, it becomes clear that only the pions contribute to the leading correction in $\langle M^2\rangle$. Therefore it suffices to prove the theorem for a nonlinear model defined for an N-component field $\mathbf{S}(x)$ constrained by $\mathbf{S}^2(x)=1$. For such a system it has been known for a while that $O(N)$-invariant correlation functions, evaluated in an infinite volume at nonexceptional momenta, are infrared finite.[6] Therefore, in a finite volume, $\Delta_s(r)$ approaches a finite limit for $2\pi r_\mu/L=k_\mu\neq 0$ kept fixed. In four dimensions the infrared behavior is softened by two powers of momenta in the integration measure for each loop and therefore the approach is faster than $1/L^2$. The most general expression compatible with $\mathbf{S}^2(x)=1$ is

$$G_s(x)=1+L^{-4}\sum_{r\neq 0}\frac{\exp(2\pi i r\cdot x/L)-1}{A_L(r)},$$

where $A_L(r)$ approaches $B_\infty(k)$ faster than $1/L^2$ for $2\pi r/L=k\neq 0$. By definition we have $z^2(N-1)=1/b_0$ where $B_\infty(k)\approx b_0k^2$ for small k. In a finite volume we now obtain

$$L^{-4}\sum_x G_s(x)$$
$$=1-\int_k 1/B_\infty(k)-\alpha/b_0L^2+\text{lower order},$$

establishing the result. The fluctuating length in the linear model can only rescale both the leading and the subleading terms by the same amount and thus the theorem holds there too.

Formula (18) could also be checked numerically, either by the method of Ref. 1 or by investigation of the shape dependence which affects the $1/L^2$ correction but not the leading term. The shape dependence enters through α by some appropriate rescalings of the components of the vectors n summed over in (16). It might be useful to calculate α for lattices corresponding to Lie algebras with larger Weyl groups[7] (in four dimensions the natural choice is associated with F_4) where periodic boundary conditions are more complicated. Such lattices hold the promise of being closer to being rotationally invariant. To know that formula (18) is correct and useful in practice would be very useful in general: For example, the universality of c could be exploited also in QCD simulations.

The main new point of the present paper is to propose exploitation of Eq. (7) for measuring f_π. Some variants of Eq. (7) have been proposed before in other contexts. Fisher and Privman[8] write similar expressions using instead of f_π and z a helicity modulus[9] and infinite-volume magnetization. These parameters are related to ours; the relations are simple at $N=\infty$, but for finite N some group-theory factors may enter and these the author has not yet worked out. Fisher and Privman used a phenomenological model and checked some of their results for the spherical model which corresponds to the $N=\infty$ limit. They also conjectured a formula which might work for arbitrary N.

In particle physics, phenomenological Lagrangeans in the context of soft-pion physics have a very precise meaning.[10] Finite-volume effects in QCD-related applications have been recently studied by Gasser and Leutwyler using such phenomenological Lagrangeans.[11] It would be nice if one could "go back to basics" and establish independently the validity of the usage of phenomenological Lagrangeans, the same way that this was done in the past for scattering amplitudes,[3,12] but now for the purpose of calculating leading finite-volume effects. There are some problems in doing this: It is somewhat unclear whether one has to incorporate explicit, subleading, size dependences in the effective Lagrangean. Nevertheless, the author is guessing that some "finite-size theorems," on the same level of rigor as the "soft-pion theorems" of the late '60's and early '70's, are provable.

The technical approach for the calculation of the $1/N$ and $1/L$ expansions as well as the idea to use one to find out about the other was mainly based on earlier papers by Brézin and collaborators.[13] Brézin and Zinn-Justin have also investigated $1/L^2$ corrections using a renormalization-group approach combined with an expansion around two dimensions.

In general, it is the author's conviction that there is a lot left to be gained when doing particle-physics-oriented Monte Carlo simulations by paying more attention to boundary conditions and boundary effects. For example, in the problem at hand, the next correction to Eq. (7) may give us a handle on the $\sigma\pi\pi$ coupling. The analysis of this correction would become a necessity should the logarithmic term in (18) turn out to be nonnegligible in practice. In a different but related context, one should also keep in mind some beautiful results obtained by Lüscher.[14]

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Note added.—Some recent developments that took place after the circulation of the original preprint form of the present paper ought to be mentioned: (a) The method suggested in this paper has been successfully implemented resulting in a very accurate and computer-time efficient evaluation of f_π.[15] (b) Gasser and Leutwyler have recently provided a more rigorous foundation for their effective-Langrangean technique,[16] and this provides another (indirect) proof for the validity of Eq. (18). (c) The constant α has been calculated for F_4 lattices.[17]

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