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Anderson Localization for a Two-Dimensional Rotor

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A two-dimensional generalization of the one-dimensional kicked-rotor model is introduced. As in one dimension, this model can be mapped on a two-dimensional Anderson model for localization. It is found that all the states are localized in momentum space and that the localization length grows exponentially with the mean free path, as expected from the scaling theory for localization. This suggests that the correspondence between the quantum dynamics of chaotic systems and Anderson localization that was found in one dimension holds in two dimensions as well.

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Quantum mechanical systems that are chaotic in the classical limit have been studied extensively in recent years.[1,2] In particular, it was found that quantal effects tend to suppress classical chaos.[3] For a certain class of simple one-dimensional kicked rotors it was shown that this suppression results in localization in angular momentum.[4-6] This localization is of a similar orgin to that of Anderson localization of electrons on disordered lattices.[6] In the framework of the theory of Anderson localization[7] the behavior of generic systems is determined by their dimensionality d: For $d \leq 2$ all the electronic states are exponentially localized, while for $d > 2$ there are extended as well as localized states. For the marginal dimensionality $d = 2$ the localization length grows exponentially with the mean free path. It is natural to investigate whether the correspondence[6] between the quantal behavior of systems that are chaotic in the classical limit and the Anderson-localization problems also holds for dimensionalities higher than one. In this Letter we will show that for a two-dimensional generalization of the kicked rotor[4-6] localization takes place in angular momentum space and the localization length grows exponentially with the mean free path. This problem is of great fundamental interest for the understanding of the correspondence between classical and quantum mechanics.[1,2] It is also of experimental relevance for molecular[8] and atomic[9] beam experiments and for the understanding of mesoscopic systems.[10]

We will study the system that is defined by the Hamiltonian[11]

$$H = H_0(\hat{n}_1, \hat{n}_2) + V(\theta_1, \theta_2) \sum_m \delta(t - m), \tag{1}$$

where

$$H_0(\hat{n}_1, \hat{n}_2) = \tfrac{1}{2} [\tau_1 \hat{n}_1^2 + \tau_2 \hat{n}_2^2], \tag{2}$$

while $\hat{n}_1$ and $\hat{n}_2$ are the momenta that are conjugate to the angles θ_1 and θ_2 (the quantum operators are $\hat{n}_1 = -i\partial/\partial\theta_1$ and $\hat{n}_2 = -i\partial/\partial\theta_2$). This Hamiltonian describes a system that is kicked periodically and is integrable in the absence of the kicking potential. The units of time are chosen so that the period between the kicks is unity. In what follows, the driving potential

$$V(\theta_1, \theta_2) = k \cos\theta_1 \cos\theta_2 \tag{3}$$

will be used. This is a natural generalization of the one-dimensional model[4-6] with $H_0 = \frac{1}{2} \tau_1 \hat{n}_1^2$ and $V(\theta_1) = k \cos\theta_1$. In the classical limit the motion that is generated by the Hamiltonian (1) is chaotic and for a sufficiently strong driving potential there is diffusion in phase space.[12] In this aspect its classical behavior is similar to that of the corresponding one-dimensional problem.

The quantal behavior is determined by the Schrö-

dinger equation $i\partial\psi/\partial t = H\psi$ (in units where Planck's constant is $\hbar = 1$, which will be used in what follows). In one dimension it was found that all the quasienergy states are localized if τ_1/π is a generic irrational number. These states satisfy an equation that is similar to a tight-binding model. In order to obtain a similar equation for (1), let us look on a specific quasienergy state $\psi_\omega = e^{-i\omega t}u_\omega(\theta_1,\theta_2,t)$, where $u_\omega(\theta_1,\theta_2,t) = u_\omega(\theta_1,\theta_2,t+1)$. Let u^- and u^+ be the values of u_ω just before and after a kick, respectively. It is convenient to define $\bar{u} = (u^+ + u^-)/2$ and its Fourier components

$$u_{\mathbf{n}} = \langle n_1 n_2 | \bar{u} \rangle = \int\int_0^{2\pi} \frac{d\theta_1\, d\theta_2}{2\pi} e^{-i(n_1\theta_1 + n_2\theta_2)} \bar{u}(\theta_1,\theta_2), \tag{4}$$

where $\mathbf{n} = (n_1, n_2)$ is a two-dimensional vector. These Fourier components then satisfy the tight-binding equation

$$T_{\mathbf{n}} u_{\mathbf{n}} + \sum_{\mathbf{r}\neq 0} W_{\mathbf{r}} u_{\mathbf{n}+\mathbf{r}} = E u_{\mathbf{n}}, \tag{5}$$

where

$$T_{\mathbf{n}} = \tan\tfrac{1}{2}(\omega - E_{\mathbf{n}}), \tag{6}$$

with

$$E_{\mathbf{n}} = \tfrac{1}{2}(\tau_1 n_1^2 + \tau_2 n_2^2). \tag{7}$$

The hopping-matrix elements are

$$W_{\mathbf{r}} = \langle \mathbf{r} | W(\theta_1,\theta_2) \rangle = \int\int_0^{2\pi} \frac{d\theta_1\, d\theta_2}{2\pi} e^{-i(r_1\theta_1 + r_2\theta_2)} W(\theta_1,\theta_2), \tag{8}$$

where

$$W(\theta_1,\theta_2) = \tan[\tfrac{1}{2} V(\theta_1,\theta_2)], \tag{9}$$

and $E = -W_0$. If τ_1/π and τ_2/π are rational, then $\{T_{\mathbf{n}}\}$ is periodic and (5) describes a periodic solid whose eigenstates are the Bloch states. For the model (1) these eigenstates correspond to the quantum resonances. If only τ_1/π or τ_2/π is rational, the states are localized in one direction in momentum space and extended in the other one. If both τ_1/π and τ_2/π are generic irrational numbers, $\{T_{\mathbf{n}}\}$ behaves like a pseudorandom sequence. The justification is similar to the one that was used in Ref. 6. For the one-dimensional model that corresponds to (1), it was found[6] that the sequence $\{T_n\}$ is sufficiently pseudorandom for localization to take place. However, it should be mentioned that the localization length is equal to that of the corresponding model where $\{T_n\}$ is truly random only if $V(\theta_1)$ is such that the hopping matrix elements $W_{\mathbf{r}}$ do not vanish for nearest neighbors only. For the potential $V(\theta_1) = k\cos\theta_1$ the localization length exhibits systematic deviations from that of the corresponding random system.[13]

As for the one-dimensional case, $W(\theta_1,\theta_2)$ is singular for $k > \pi$. Also for the two-dimensional case this singularity can be avoided by a mapping on a tight-binding model with diagonal and off-diagonal pseudorandom matrix elements that are correlated. The derivation that was introduced by Shepelyansky[14] for the one-dimensional problem can be easily generalized to the present problem.

The dynamics of a system that is described by the tight-binding model (5) is determined by the hopping $W_{\mathbf{r}}$, that tends to delocalize the particle, and by the diagonal energies $T_{\mathbf{n}}$, that tend to localize it. If $\{T_{\mathbf{n}}\}$ is truly random, the scaling theory for localization predicts that in two dimensions all the states are localized.[7] Moreover, the localization length ξ grows exponentially with the mean free path l, namely $\xi = a\exp(\bar{b}l)$, where a and $\bar{b}$ are constants. In order to calculate the mean free path, it is convenient to use the single-period evolution operator that propagates the wave function from a time just before a kick to the time just before the next one. For the Hamiltonian (1) it takes the form

$$\hat{U} = e^{-iH_0(\hat{n}_1,\hat{n}_2)} e^{-iV(\theta_1,\theta_2)}. \tag{10}$$

The matrix elements in the momentum representation for the driving potential (3) are

$$U_{\mathbf{n},\mathbf{n}+\mathbf{r}} = e^{-iE_{\mathbf{n}}}(-i)^{r_1} J_p(k/2) J_q(k/2)[1 + (-1)^{r_1+r_2}]/2, \tag{11}$$

where $J_p(k/2)$ and $J_q(k/2)$ are Bessel functions of the first kind of the orders $p = (r_1 + r_2)/2$ and $q = (r_1 - r_2)/2$. The factor $\exp[-iV]$ leads to transitions between different momentum states and is related to the hopping-matrix elements in (5) via the relation (9). The free-motion part $\exp[-iH_0]$ is related to the diagonal energies $T_{\mathbf{n}}$ of the tight-binding model (5). The assumption that $\{T_{\mathbf{n}}\}$ is random is equivalent to the assumption that the phase factor $\exp[-iE_{\mathbf{n}}]$ in (11) is random. Consequently between the kicks the phase of the wave function is randomized. The resulting mean free path l satisfies

$$l^2 = \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} (r_1^2 + r_2^2) |U_{0,\mathbf{r}}|^2 = \frac{k^2}{2}, \tag{12}$$

where the explicit form (11) was used. If one assumes that $\{T_{\mathbf{n}}\}$ is random, the scaling theory for localization implies

$$\xi = a e^{bk}, \tag{13}$$

where $b = \bar{b}/\sqrt{2}$.

In order to test whether the localization theory really applies to the model (1) the evolution of a wave function that is initially localized on a definite angular momentum state was investigated. For this purpose the wave function was propagated with the evolution operator (10) starting from the initial state $\mathbf{n}=0$. The calculations were performed for various values of k, τ_1, and τ_2. For generic values of τ_1 and τ_2 we found that after some time the wave function ceases to spread in momentum space, and its amplitude falls off exponentially, at a rate that is independent of time. The amplitudes of the various angular momentum states oscillate in time but their magnitude is determined by an exponential envelope in momentum space. The typical behavior of the wave functions is depicted in Fig. 1. The localization length is found from a least-squares fit of an exponential falloff to these wave functions. As one can see, the wave functions are isotropic in momentum space. We analyzed the oscillations in time of an amplitude at fixed angular momentum and found that they are quasiperiodic. This reflects the fact that the quasienergy states are localized and only a few of them have considerable amplitude on a given momentum state. The localization length as a function of k for $\tau_1=1$ and $\tau_2=\sqrt{2}$ is presented in Fig. 2. The numerical results are in agreement with the prediction (13) of the localization theory with $a=0.46$ and $b=0.93$. Different values of τ_1 and τ_2 lead to different values of a and b. The numerical data rule out a power-law dependence of ξ on k. For rational τ_1/π or τ_2/π the wave functions are extended. After several time steps they spread on all the states of the basis that is used in the calculation, as expected. There are special values of τ_1/π and τ_2/π for which this spreading does not take place because of an infinite degeneracy[15] of the extended states.

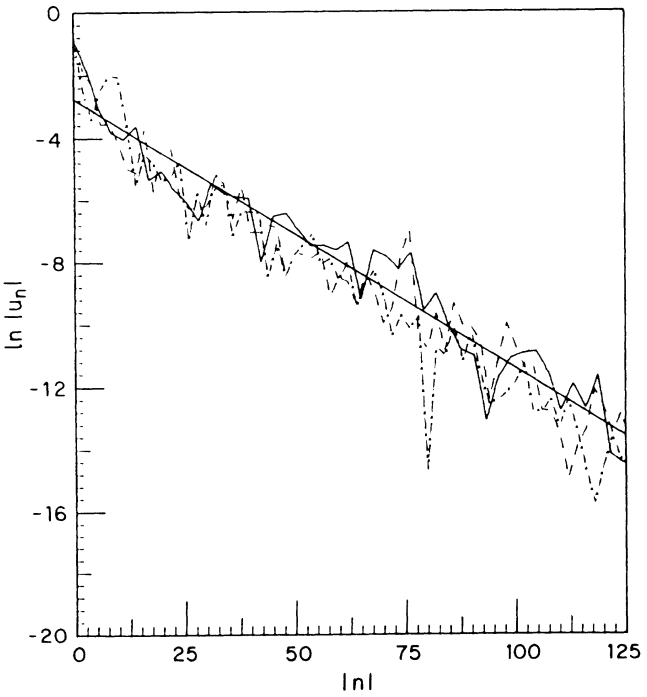

FIG. 1. Logarithm of the absolute value of the wave function for $k=3.5, \tau_1=1$, and $\tau_2=\sqrt{2}$ after $m=1023$ time steps as a function of momentum along the n_1 axis (dot-dashed line), the n_2 axis (dashed line), and the diagonal (solid line). The initial wave function is localized on $\mathbf{n}=0$. The straight line is the best fit for exponential decay of the wave function along the diagonal.

The main limitation of the numerical calculations is the finite size of the basis that can be used. We used a lattice of $256\times256=65\,536$ states in momentum space. The deviations from a straight line in Fig. 2 when $\xi\geq100$ result from the finiteness of the basis and therefore (13) was verified numerically only for $k\leq5$. Note that the evolution operator (11) is a $65\,536\times65\,536$ matrix for this basis. The operator $\hat{U}$ was applied in two steps. First $\exp[-iV(\theta_1,\theta_2)]$ was applied in the θ representation, and then $\exp[-iH_0(n_1,n_2)]$ in the n representation. The transformation between the representations was performed by the fast-Fourier-transform method. This method reduces the number of multiplications compared to a direct multiplication by the matrix (11) approximately by a factor of 10000. The evolution of the wave function was followed for 1023 and 2047 time steps in order to verify the stabilization of the envelope of the wave function. The quasiperiodic behavior of the wave function was obtained with the help of its Fourier transform in time.

In summary, we have shown that for the two-dimen-

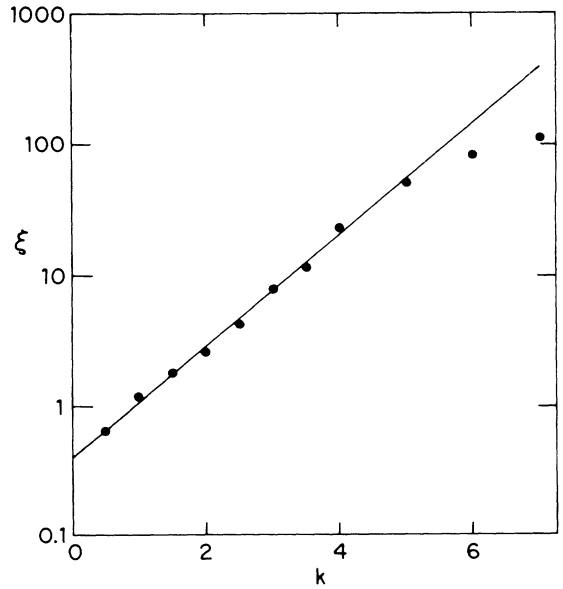

FIG. 2. Localization length ξ as a function of k. The numerical results are denoted by circles. The line is the best fit for exponential growth. For this calculation we used $\tau_1=1$ and $\tau_2=\sqrt{2}$.

sional rotor (1) all the states are localized and the localization length increases exponentially with k. This is in agreement with the scaling theory for localization and supports the correspondence between the quantal behavior of classically chaotic systems that are defined by Hamiltonians like (1) and Anderson localization on disordered lattices, that is described by tight-binding models like (5). In the present work this correspondence was established for the first time (to our knowledge) for two-dimensional systems. If it holds in three dimensions as well, one expects to find there a transition between localized and extended states. This problem should be investigated in the future.

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