Fractional Angular Momentum and Magnetic-Flux Quantization

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It is shown that the fractional angular momentum of charged particles orbiting around a magnetic-flux tube would result in fractional magnetic-flux quantization. A crucial experiment to test for the fractional quantum of magnetic flux as well as anyons is proposed and analyzed.

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The fact that charged particles orbiting around a magnetic-flux tube with flux Φ have an orbital angular momentum $(\text{integer}+q\Phi/ch)\hbar$ was known long ago.[1] On the basis of a similar idea it was stated recently that the flux-tube–particle composites called anyons have fractional angular momentum.[2,3] Exotic statistics pertaining to n identical anyons was also developed.[4] A controversy still exists,[5,6] since there is a very fundamental distinction between the canonical angular momentum and the kinetic angular momentum which relates to the energy. In the theory of anyons, however, the difference between the canonical angular momentum eigenvalue and the kinetic angular momentum eigenvalue is removed formally by a singular gauge transformation[3,4] and physically by the discard of the return flux of the magnetic-flux tube.[7]

Specifically, in the case of anyons, fractional angular momentum (implying the canonical angular momentum hereafter) might be possible. Then the fractional angular momentum would result in very unusual phenomena, and it is of great interest to search for anyons as a practical matter. We now explore a possibility relating the fractional angular momentum to magnetic-flux quantization.[8] It is shown that the magnetic-flux quantization must be consistent with the angular momentum quantization of superelectrons. If the theory of anyons is applied to a system consisting of an inaccessible magnetic flux and superelectrons in a superconducting hollow cylinder, the fractional angular momentum of the superelectrons leads necessarily to a fractional quantum of magnetic flux trapped in the hollow cylinder. The result contradicts a common belief that the flux trapped in the hole of a multiply connected superconducting body must be an integer multiple of a quantum unit. It is also pointed out that the possibility of fractional flux quantization cannot be rejected on the basis of previous experiments[9-12] which confirm the usual flux quantization, and a special kind of experiment is needed to resolve the contradiction.

In the present note we start from the Ginzburg-Landau (GL) theory to derive a general relation between the angular momentum quantization of superelectrons and the magnetic-flux quantization. An experiment to test the trapped fractional quantum of magnetic flux as well as anyons is proposed and analyzed.

The order parameter ψ in GL theory is considered as the coherent wave function which, roughly speaking, corresponds to the center-of-mass wave function of BCS pairs. One therefore can compare with the single-particle Schrödinger equation to define the single-particle-like angular momentum of the superelectrons. The total magnetic flux of the applied magnetic field and the field due to the superelectrons can be explicitly calculated.

The free energy of a superconducting state is given by

$$F_s=F_n+\int\left\{-\alpha|\psi|^2+\frac{\beta}{2}|\psi|^4+\frac{H^2}{8\pi}+\frac{1}{2\mu^*}\left[\psi^*\left(-i\hbar\nabla-\frac{e^*}{c}\mathbf{A}(\mathbf{x})\right)^2\psi\right]\right\}d^3x, \tag{1}$$

where F_n is the free energy of the normal state, $|\psi(\mathbf{x})|^2=n_s(\mathbf{x})$ is the density of superelectrons; $\mu^*=2\mu$ and $e^*=2e$ are the mass and charge of a bound pair, respectively.

Variation of F_s with respect to ψ^* and $\mathbf{A}$ yields self-consistent equations

$$\frac{1}{2\mu^*}\left(-i\hbar\nabla-\frac{e^*}{c}\mathbf{A}\right)^2\psi-(\alpha-\beta|\psi|^2)\psi=0, \tag{2}$$

$$\nabla^2\mathbf{A}=-(4\pi/c)\mathbf{J}_s. \tag{3}$$

The boundary condition on the surface of the specimen is

$$\frac{\partial[-i\hbar\nabla\psi-(e^*/c)\mathbf{A}\psi]}{\partial\hat{\mathbf{n}}}=0, \tag{4}$$

where $\hat{\mathbf{n}}$ is the normal unit vector to the surface. The supercurrent density is defined by

$$\mathbf{J}_s=-\frac{ie^*\hbar}{2\mu^*}(\psi^*\nabla\psi-\psi\nabla\psi^*)-\frac{e^{*2}}{\mu^*c}|\psi|^2\mathbf{A}. \tag{5}$$

Let us consider a two-dimensional multiply connected superconductor with a hole Δ as shown in Fig. 1. The fluxoid embraced by a closed contour C is defined by London[8] as

$$\Phi_L\equiv\oint_c\mathbf{A}\cdot d\mathbf{l}+\frac{\mu^*c}{e^{*2}}\oint\frac{\mathbf{J}_s}{n_s}\cdot d\mathbf{l}. \tag{6}$$

By means of

$$\mathbf{J}_s=e^*n_s\mathbf{v},\quad \mathbf{v}=(\mu^*)^{-1}[\mathbf{P}-(e^*/c)\mathbf{A}], \tag{7}$$

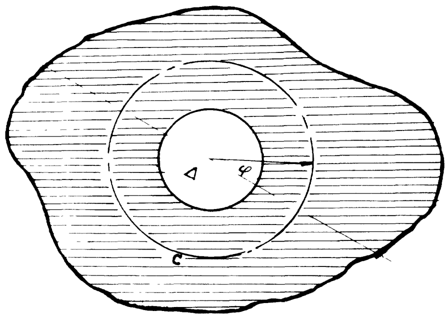

FIG. 1. Multiply connected superconductor.

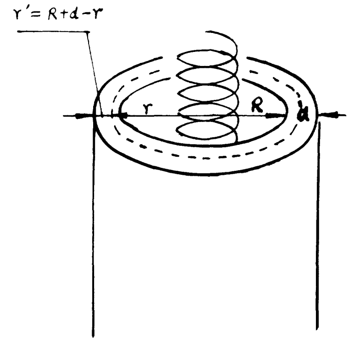

FIG. 2. Experiment testing the fractional angular momentum and fractional quantum of magnetic flux.

the fluxoid then is

$$\Phi_L=(c/e^*)\oint_c \mathbf{p}\cdot d\mathbf{l}=(c/e^*)\oint p_\phi d\phi. \tag{8}$$

The quantization of the fluxoid is equivalent to the Bohr-Sommerfeld quantization condition, where p_ϕ is the angular momentum of the superelectrons; ϕ is the azimuthal angle. If there exists an external magnetic flux $\Phi^{\rm ext}=\delta\Phi_0^*$ $(0<\delta<1,\ \Phi_0^*\equiv ch/e^*=\Phi_0/2;\ \Phi_0\equiv ch/e)$ inaccessible to superelectrons through the hole, the fractional angular momentum quantization of superelectrons

$$\oint_c P_\phi d\phi=h(\text{integer}+\delta), \tag{9}$$

leads, of course, to a fractional quantum of fluxoid:

$$\Phi_L=(\text{integer}+\delta)\Phi_0^*. \tag{10}$$

In practical experiments the total magnetic flux trapped in a superconducting hollow cylinder is different from the London fluxoid except that $\mathbf{J}_s=0$ in the cylinder. To obtain the trapped magnetic flux, Eqs. (2) to (5) should be solved in the cylindrical geometry.

An experiment to test the fractional angular momentum can be prepared as shown in Fig. 2. An infinitely long, thin solenoid (in practice the length of the solenoid is much longer than its radius) coincides with the axis of a superconducting hollow cylinder. One might establish a stable magnetic flux $\delta\Phi_0^*$ through the solenoid above the transition temperature T_c. After the induced current dies off in the cylinder ($T>T_c$ in normal state), one cools the system below the transition temperature T_c.

Now let us calculate the magnetic flux trapped in the cylinder. In the presence of an external magnetic flux confined in the solenoid, the superelectrons are subject to an azimuthal vector potential

$$A_\phi^{\rm ext}=\delta\Phi_0^*/2\pi r. \tag{11}$$

The total azimuthal vector potential is

$$A_\phi=A_\phi^{\rm ext}+A_\phi^s, \tag{12}$$

where A_ϕ^s states the vector potential due to superelectrons. According to Wilczek[3] and Wu,[4] the external vector potential of inaccessible magnetic flux can be eliminated by a gauge transformation for convenience[13]:

$$A_\phi'=A_\phi-\frac{1}{r}\frac{\partial}{\partial\phi}\Lambda=A_\phi^s,\quad \Lambda=\frac{\delta\Phi_0^*}{2\pi}\phi. \tag{13}$$

The required Λ is, however, not a well-defined (2π periodic) function of the angle ϕ. This fact reflects itself in the transformation of the wave function

$$\psi'(r,\phi)=\exp(i\delta\phi)\psi(r,\phi). \tag{14}$$

In GL theory all the "superelectrons" are described by the same, coherent, wave function ψ which is single valued. The mth eigenfunction of angular momentum is

$$\psi_m=R_m(r)e^{im\phi}, \tag{15}$$

where m is the integer. The allowed angular wave functions therefore have the fractional spectrum of angular momentum

$$p_\phi=(\text{integer}+\delta)\hbar. \tag{16}$$

The substitution of A_ϕ' and $\psi'(r,\phi)$ into Eqs. (2) and (3) yields the equation for $R_m(r)$ and the equation for $A_\phi^s(r)$:

$$d^2R_m/dr^2+r^{-1}dR_m/dr-[(m+\delta)/r-(2\pi/\Phi_0)A_\phi^s(r)]^2R_m+(2\mu^*/\hbar^2)(\alpha-\beta|R_m|^2)R_m=0, \tag{17}$$

$$(d^2/dr^2+r^{-1}d/dr-r^{-2})A_\phi^s(r)=-(4\pi/c)J_\phi^s(r), \tag{18}$$

where the azimuthal supercurrent density is

$$J_\phi^s(r)=-[(e^*\hbar/\mu^*)(m+\delta)/r-(e^{*2}/\mu^*c)A_\phi^s(r)]\,|R_m(r)|^2. \tag{19}$$

Exact solutions of $R_m(r)$ and $A_\phi^s(r)$ satisfying the boundary condition (4) are

$$R_m=(\alpha/\beta)^{1/2} \tag{20}$$

and

$$A_\phi^s(r)=[(m+\delta)/2\pi r]\Phi_0^*. \quad (21)$$

The new consequence from the theory of anyons is that the fractional angular momentum of the superelectrons results in an irrotational vector potential (21) ($\nabla\times\mathbf{A}^s=0$) which leads to neither magnetic field nor supercurrent in the superconducting cylinder. Therefore the external magnetic flux in the solenoid remains and the total magnetic flux trapped in the superconducting cylinder is equal to the London fluxoid

$$\Phi_L=\oint \mathbf{A}^s\cdot d\mathbf{l}=(m+\delta)\Phi_0^*. \quad (22)$$

Since the magnetic flux $\delta\Phi_0^*$ in the solenoid is established above the transition temperature T_c, there is no induced electric field applied to the superelectrons which appear when the temperature is below T_c. The solutions (20) and (21) with a zero supercurrent density corresponding to the state of the lowest energy are physically reasonable.

On the contrary, if the angular momentum of the superelectrons is quantized in the conventional way, the applied magnetic flux in the solenoid would induce a net supercurrent in the superconducting cylinder. Solving the set of nonlinear equations (2) to (5) with perturbation theory, it is easy to obtain the distributions of magnetic field and supercurrent density in the cylinder[14]:

$$H_Z(r)=(\delta\Phi_0^*/\pi R^2)D^{-1}[K_0(\eta_2)I_0(r/\lambda_L)-I_0(\eta_2)K_0(r/\lambda_L)]; \quad (23)$$

$$J_\phi^s(r)=(-\delta\Phi_0^*c/4\pi^2R^2D\lambda_L)[K_0(\eta_2)I_1(r/\lambda_L)+I_0(\eta_2)K_1(r/\lambda_L)]. \quad (24)$$

Here I and K are the Bessel functions of imaginary argument,[15]

$$D\equiv I_0(\eta_2)K_2(\eta_1)-K_0(\eta_2)I_2(\eta_1),$$

and

$$\eta_1=R/\lambda_L,\quad \eta_2=(R+d)/\lambda_L; \quad (25)$$

R $(R+d)$ is the inner (outer) radius of the cylinder, while

$$\lambda_L=[\mu^*c^2/4\pi e^{*2}n_s^{(0)}]^{1/2} \quad (26)$$

is the London penetration depth, $n_s^{(0)}$ being the density of superelectrons in the absence of magnetic field. The total magnetic flux trapped in the cylinder can be calculated and is equal to[14]

$$\Phi_T=m\Phi_0^*+\delta\Phi_0^*\{1+D^{-1}[K_0(\eta_2)I_0(\eta_1)-I_0(\eta_2)K_0(\eta_1)] + (2\eta_2/D\eta_1^2)[K_0(\eta_2)I_1(\eta_2)-I_0(\eta_2)K_1(\eta_2)]+(2/D\eta_1)[I_0(\eta_2)K_1(\eta_1)-K_0(\eta_2)I_1(\eta_1)]\}. \quad (27)$$

In practical experiments, R and d are much greater than $\lambda_L\sim5\times10^{-8}$ m. If we take the large-argument limit of the Bessel functions in Eq. (27), $\eta_2>\eta_1\gg1$, the total magnetic flux trapped in the cylinder is approximately quantized:

$$\Phi_T\sim(m+2\delta/\eta_1)\Phi_0^*\sim m\Phi_0/2. \quad (28)$$

On the basis of the phenomenological theory of superconductivity, we have shown that when an inaccessible magnetic flux $\delta\Phi_0/2$ exists, the magnetic flux trapped in a superconducting hollow cylinder is quantized either as $(\text{integer}+\delta)\times\Phi_0/2$ or $(\text{integer})\times\Phi_0/2$ depending on the quantization condition of angular momentum. One might think that the fractional flux quantization would be rejected theoretically by the requirement of low energy of the electron pairs in BCS theory. However, the fact that fractional angular momentum quantization results necessarily in the fractional magnetic-flux quantization is even more obvious from the viewpoint of the BCS theory. To see this, let us follow Schrieffer[16] to give a simple argument for the fractional magnetic-flux quantization. For a macroscopic cylinder we have that $R\gg\xi$ and $d\gg\lambda_L$, ξ being the coherent length while λ_L is the penetration depth. In the cylinder, electrons on a ring with radius r ($r-R\gg\lambda_L$; $R+d-r\gg\lambda_L$) satisfy, in zero-order perturbation, the Schrödinger equation

$$(-\hbar^2/2\mu r^2)(d/d\phi-i\gamma)^2\psi_M=E_M\psi_M, \quad (29)$$

where μ is the mass of an electron and $\gamma\Phi_0$ is the magnetic flux embraced by the hollow cylinder. The spectrum of energy is

$$E_M=(\hbar^2/2I)(M-\gamma)^2. \quad (30)$$

The low-energy state for the paired system requires[16]

$$M-\gamma=-(\overline{M}-\gamma). \quad (31)$$

If the fractional angular momentum quantization due to the external magnetic flux ($\Phi^{\text{ext}}=\delta\Phi_0/2$), namely,

$$P_\phi=M\hbar=(m+\tfrac{1}{2}\delta)\hbar, \quad (32)$$

m being an integer, is allowed, from (31) one immediately obtains the fractional magnetic-flux quantization:

$$\begin{aligned}\gamma&=(\delta+\text{integer})/2,\\ \Phi&=\gamma\Phi_0=\Phi^{\text{ext}}+\text{integer}\times\Phi_0/2.\end{aligned} \quad (33)$$

The total flux takes, of course, discrete values, which differ by integer multiples of $\Phi_0/2$, but start from Φ^{ext}.

Let us now see how an experiment can test the fractional quantum of magnetic flux. This is evident from what we have said above. One must build a very well-shielded solenoid in such a way that there is practically no leakage of magnetic field. At room temperature a flux Φ^{ext} which differs from $m\Phi_0/2$ is established through the solenoid. One cools the system below the transition temperature of the superconductor, taking care not to change the current in the solenoid, and then measures the total flux across the superconductor with the aid of electron interferometer or SQUID. If this value remains equal to Φ^{ext}, one obtains the fractional quantum of magnetic flux as well as the anyons. If one changes now continuously the current in the solenoid, one will observe the usual jumps of $\Phi_0/2$ in the total flux. But an important thing is that the absolute value of the flux is shifted with a common amount $\Phi^{ext}=\delta\Phi_0/2$.

There are two new points in the experiment to test anyons: (1) A nonlocal applied magnetic flux, namely, the flux confined in a well-shielded solenoid is needed. (2) The absolute value of the trapped magnetic flux should be measured. In early experiments[9–12] superconducting cylinders were cooled below the transition temperature in the presence of an axial, uniform magnetic field which, however, is local to the superelectrons. It has been shown that[14] the shielding current is necessarily induced by the applied magnetic field and the total trapped magnetic flux is quantized in the usual way that $\Phi_T \sim m\Phi_0/2$. Recently the solenoid is used as a source of applied magnetic field.[17] But the experiment is arranged to measure the usual jumps of magnetic flux due to the variation of the applied magnetic field in the solenoid.

In conclusion, the fractional angular momentum of charged particles orbiting around a magnetic-flux tube would lead to the fractional magnetic-flux quantization which has not yet been observed. The experiment proposed in the present paper offers a possibility to test anyons and fractional quanta of magnetic flux.

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