Truncated-Fractal Basin Boundaries in Forced Pendulum Systems

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(Received 4 May 1987)

This Letter reexamines questions about the practicality of computing the *exact* stability domains of certain classes of pendulum systems. We display plots showing that the intertwined neighboring domains have a self-affine truncated-fractal structure. A simple proof of the existence of diffeomorphisms from connected basins to striated basins is also presented.

PACS numbers: 03.20.+i, 05.45.+b

Reported here is the study of truncated-fractal deformations in the stability domains of transiently driven, damped pendulum systems. In systems with constant driving functions, the attractors vary from simple limit cycles to chaotic attractors of immense complexity. The forcing function of the pendulum system studied here has a nonautonomous sinusoidal driving term which exponentially decays to zero. It is intuitively clear that this system, after an appreciable length of time, resembles an autonomous, dissipative pendulum with asymptotically stable fixed points separated by intervals of 2π.

Damped, driven pendulum systems have been used successfully, in the past, to model complicated behavior in[1-5] nonlinear systems. We choose to study the transiently forced class of pendulums and the distinguishing feature of our investigation is that we shift our emphasis from the structural complexity of the attractors to the composition of the boundaries separating the domains of attraction of the various (asymptotically stable) fixed points.

The study of deformed basin boundaries reported here for the transiently driven, damped pendulum system was also motivated by the following observation that implies an apparent contradiction in theory (i.e., Zubov boundaries). For autonomous nonlinear systems with asymptotically stable fixed points, the method of Zubov predicts the existence of smooth (with respect to continuity) boundaries of stability regions. The result due to Zubov[6] is that a (smooth) stability boundary exists and is given by $Z(x)=1$ where $Z(x)$ is the solution to the partial differential equation

$$(\partial z/\partial x)^T f(x)=H(x)[Z(x)-1][1+f^T(x)f(x)]^{1/2},$$

where

$$\dot{x}(t)=f(x(t))$$

defines the system. It can be seen, later on, that the second-order, nonautonomous, transiently forced pendulum can be written as a fourth-order autonomous system with asymptotically stable fixed points. The corollary of Zubov's construction method points out that if $f\in C^r$ the boundary is C^r. Note that, although smoothness of $f(x(t))$ mandates a smooth Zubov or exact stability boundary, the partial differential equation given above, except for pathological cases, is impossible to solve explicitly.

Numerical simulations for critical values of damping and forcing parameters yielded a basin boundary (exact stability boundary) that was fractal in nature. We were thus led to investigate the apparent contradiction between the Zubov predictions and observations via numerical experiments.

Consider the damped, transiently driven pendulum system

$$\dot{x}_1=x_2,$$

$$\dot{x}_2=-dx_2-\sin x_1+ge^{-\epsilon t}\cos\omega t,$$

where x_1, x_2, d, ϵ, ω, and g denote the angular position, the angular velocity, the system damping coefficient, the exponent for forcing decay, the forcing frequency, and the forcing magnitude, respectively. Since Zubov's method of construction works only for autonomous systems, the two-dimensional nonautonomous system is rewritten as a four-dimensional autonomous system.

$$\dot{x}_1=x_2,$$

$$\dot{x}_2=-dx_2-\sin x_1+x_3,$$

$$\dot{x}_3=-\epsilon x_3+\omega x_4,$$

$$\dot{x}_4=-\omega x_3-\epsilon x_4.$$

We conclude that the exact (or Zubov) boundary is composed of trajectories that are smooth and live in R^4, but the slice in R^2 [for which $x_3(0)$ and $x_4(0)$ are chosen to be g and 0, respectively] has a truncated-fractal structure. It is, of course, still a source of wonder as to how a smooth surface in R^4 could give rise to a contour as complex as the color plot (Fig. 5) even if it is only a slice in R^2.

For the sinusoidally forced pendulum, Gwinn and Westervelt[3] demonstrated, via simulations, a self-similar or self-affine fractal structure for the parameter set $d=0.5$, $\omega=0.66$, $g=1.48$. Our simulations superimposed an $\epsilon\neq 0$ value on the Gwinn-Westervelt critical pa-

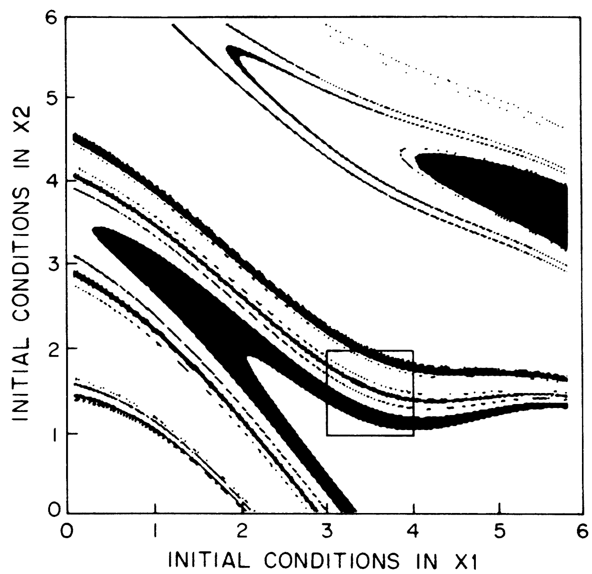

FIG. 1. Stability domain of the origin for a 350×350 initial condition grid.

rameter set and tuned ϵ to 0.014 to obtain the truncated-fractal boundary shown in Figs. 1 to 5.

Numerical integration, with use of a fourth-order fixed–step-size Runge-Kutta routine, was done for a 350×350 grid of initial conditions ranging from $x_1(0)=0$ to 2π and $x_2(0)=0$ to 6 in increments of 0.0179 and 0.0171 units, respectively. Each initial condition was integrated numerically for 60 drive cycles of ω (or 570 s of integration time) to allow the initial transients to decay. The last ten drive cycles of x_1 were averaged to determine which equilibrium point the trajectories converged to.

The set of initial conditions yielding convergent behavior in a neighborhood of the origin is shaded. The white region denotes convergence to other equilibrium points. Figure 1 shows the initial conditions that were attracted to the origin. To verify the fractal nature of the boundary (or the origin) magnifications of grid sensitivity of a portion of Fig. 1 were done and numerically integrated to obtain Fig. 2 and this was repeated recursively to get Figs. 3 and 4. The repeated birth of striations under transformations of scale can be seen in Figs. 2 and 3. Self-similarity (or more correctly self-affinity) breaks down in Fig. 4. The box dimension of the fractal was computed for the 350×350 grid to be 1.92.

Each initial condition was numerically integrated, with the same procedure, for a 750×750 grid on the Cornell National Supercomputing Facility and the results are displayed as a color plot in Fig. 5. The axis for initial conditions in pendulum angle ranges from $x_1(0)=0$ to 2π in increments of $2\pi/750$. The axis for initial conditions in pendulum velocity ranges from $x_2(0)=-6$ to 6 in increments of 12/750.

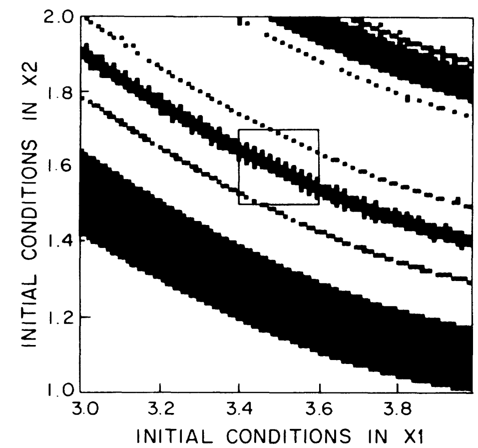

FIG. 2. Magnification of the box shown in Fig. 1.

The colors red, green, yellow, blue, magenta, cyan, and white correspond to the x_1 coordinate of the equilibrium point being -6π, -4π, -2π, 0, $+2\pi$, $+4\pi$, and $+6\pi$, respectively. Any other equilibrium point was colored black.

Note that since the $\epsilon=0$ system does not have asymptotically stable fixed points one cannot utilize Zubov theory to make predictions about the smoothness of the boundary.[7,8] On the other hand, the Poincaré map is not a useful tool in the analysis of the ϵ-positive system because the flow is not periodic.

The apparent contradiction between the smoothness of the Zubov boundary versus the self-affine fractal striations observed via numerical simulation is resolved by our showing the existence of diffeomorphisms between

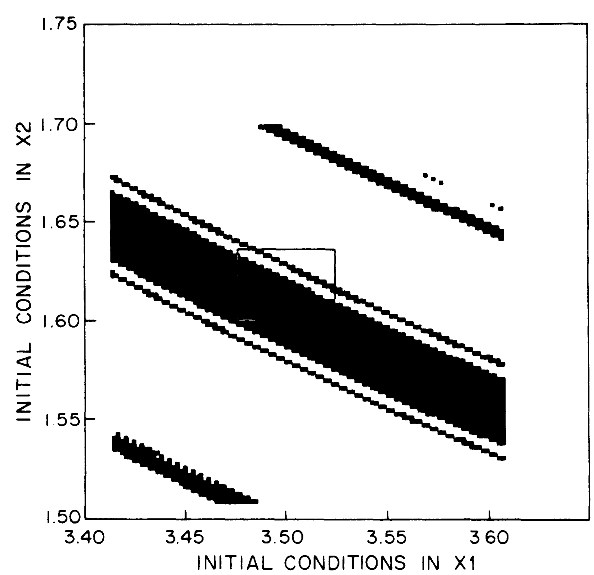

FIG. 3. Magnification of the box shown in Fig. 2.

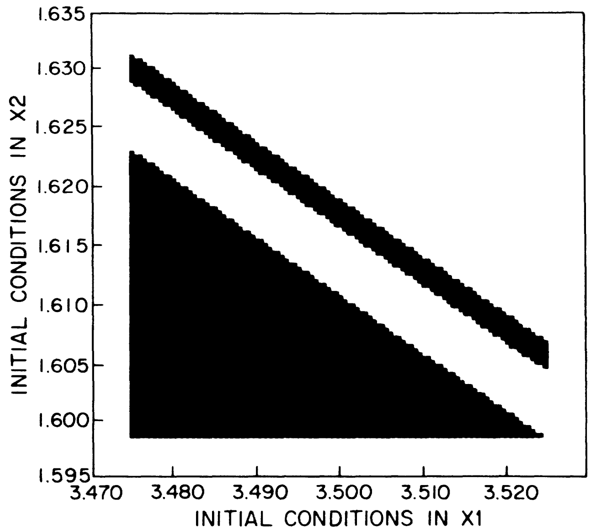

FIG. 4. Magnification of the box shown in Fig. 3.

connected and striated basins. Consider initial values of time at

$$t_0=\frac{2\pi}{\omega},\frac{4\pi}{\omega},\frac{6\pi}{\omega},\ldots,\frac{N2\pi}{\omega}\quad (N \text{ an integer}),$$

to account for the correct phasing of the $\cos(\omega t_0)$ term. Consider the driven pendulum system in a compact form

$$\ddot{x}+\dot{x}+\sin x=ge^{-\epsilon t}\cos(\omega t).$$

Let

$$h=ge^{-\epsilon t_0}.$$

Consider two such values of h: h_1,h_2 such that h_1 belongs to a parameter set for a truncated-fractal basin boundary and h_2 is sufficiently small to ensure that the forcing function perturbation does not affect the nature of the boundary.

Under operation of the flow for $t=N2\pi/\omega$ s, the basin of the system is diffeomorphic to that of a system that started at $t_0=N2\pi/\omega$ s (i.e., had a value of $h=ge^{-\epsilon N2\pi/\omega}$). In fact, formally speaking, there exist diffeomorphisms

$$\phi_h\colon B(\bar{x})^{h_1}_{t_0=0}\to B(\bar{x})^{h_2}_{t_0=N2\pi/\omega}\quad (h_2=h_1e^{-\epsilon N2\pi/\omega}),$$

by property of the flow.

Construct $[h_1,h_2]$ by choosing N sufficiently big that the domain of attraction $B(x)$ for h_1 has the same topological (Haussdorf) dimension as the domain $B(x)^{h_2}$ (that has a smooth stability boundary). This implies that, qualitatively speaking, there exist continuous deformations [at h_N:$(t_0)_N=0,2\pi/\omega,4\pi/\omega,N2\pi/\omega$] that do not tear (i.e., create new openings) or fill up existing gaps

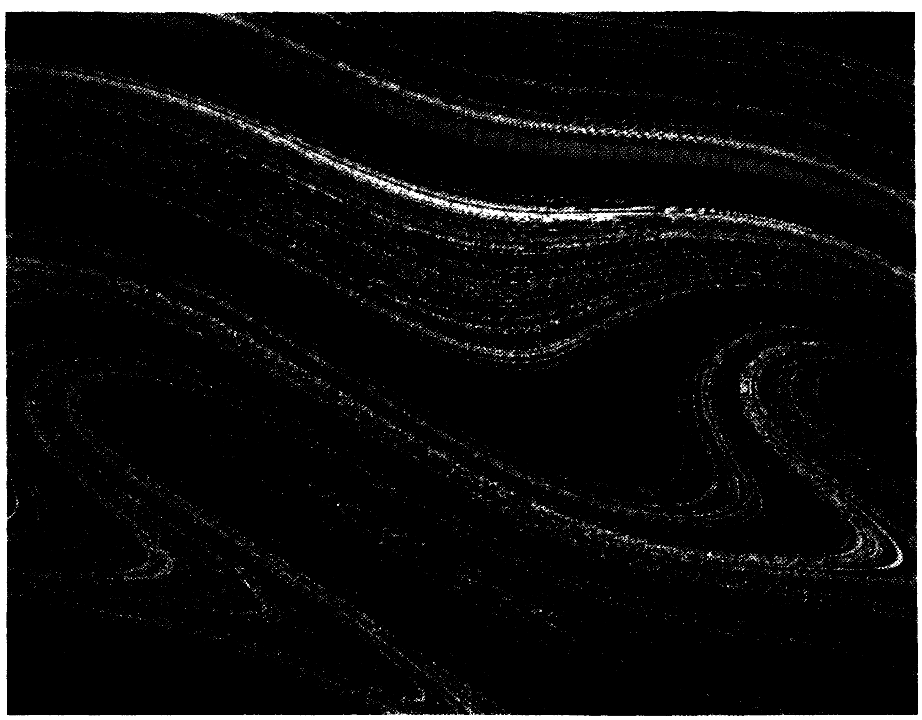

FIG. 5. Supercomputer simulations for a 750×750 grid in color. Vertical axis: initial condition in the pendulum velocity. Horizontal axis: initial condition in the pendulum angle. See text for discussion of colors.

from a truncated-fractal surface to a nonfractal one.

The new "wrinkle" that distinguishes our pendulum model from earlier studies is that the forcing function decays exponentially to zero. This implies that the attractors of our forced system are the asymptotically stable fixed points of the original unforced, damped pendulum system. We emphasize that the transiently forced system will not exhibit chaotic behavior since the forcing shrinks exponentially to zero.

It is possible that the system does exhibit transient complicated behaviors enroute to an asymptotically stable fixed point but we do not dwell on that. What is important, however, is that these simulations and theoretical results (for sufficiently small forcing)[9,10] predict the growth of intrusions of neighboring stability domains into the basin boundary of the fixed point at the origin. This raises questions of the practicality of formulating better analytic estimates for nonconservative regions of attraction in systems that implicitly fall into these classes of nonautonomous pendulums with critical driving parameters.

We thank Professor B. B. Mandelbrot (Harvard University and the IBM Research Center) for many useful discussions. This research was conducted with use of the Cornell National Supercomputing Facility, which is funded in part by the National Science Foundation, New York State, and the IBM Corporation.

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FIG. 5. Supercomputer simulations for a 750×750 grid in color. Vertical axis: initial condition in the pendulum velocity. Horizontal axis: initial condition in the pendulum angle. See text for discussion of colors.