Comment on "Space-Time as a Causal Set"

In a recent Letter by Bombelli, Lee, Meyer, and Sorkin,[1] the authors discuss the merits of viewing space-time not as a continuous manifold, but rather as a discrete set of points with connections between them defining the large-scale causal structure. This would remove the ultraviolet divergences plaguing present-day field theory, and the singularities of general relativity; it would provide a basis for making space-time, in addition to the fields existing on it, a dynamical variable[2]; and it would make the amount of information and computation going on in a finite volume of space-time finite as opposed to infinite.

This Comment is intended to explore the idea of a lattice which is Lorentz invariant, a property we know the Universe to have, at least at large scales. We know such a lattice would have to be random,[3] to avoid having preferred directions; then the local connections would have some probability distribution $\rho(x_1-x_2)=\rho(x)$. Since space-time volume $dV=d^3\mathbf{x}\,dt$ is invariant under Lorentz transformations, ρ must be a function only of Δs, the invariant interval: $\rho=\rho(\Delta s)$. Then the coordination number, or the average number of points each point is connected to, is

$$N=\int d^4x\,\rho(\Delta s). \tag{1}$$

But this integral diverges, since each hyperboloid shell of constant Δs has an infinite volume. Even if we restrict connections to the light cone, with $\rho=\rho(r)\delta(ct-r)$, Lorentz invariance requires that $\rho(r)=\text{const}/r$, and then

$$N=\int 4\pi r^2\,dr\,\rho(r) \tag{2}$$

diverges again. Besides, the next-neighbor distribution would have the form (1). Connections outside the light cone, if they exist at all, similarly diverge.

So there is no normalizable probability distribution that is Lorentz invariant: More precisely, there is no Lorentz-invariant probability measure of $\mathbb{R}^4$ besides $\delta^4(x)$.

This means that if the Universe is indeed a random lattice, the coordination number is infinite at the scale at which space-time has 3+1 dimensions and is Lorentz invariant—but of course one wants the real coordination number, at the smallest scales, to be finite. Perhaps this puzzle can be solved in terms of the 3+1 space being a "coarse graining"[1] of a lattice whose dimensionality is higher at smaller scales; the coordination number will vary in some relation to the scale of the coarse graining. However, the coordination number would seem to *decrease* with the dimensionality as we increase the scale, rather than increasing to infinity as would be needed to resolve the paradox. In the absence of such a resolution there seem to be four possibilities:

(1) Space-time is not, in fact, discrete.

(2) Space-time is discrete, but it has the nasty property that every point is influenced by an infinity of "nearest neighbors" which, in a given frame, are arbitrarily far back in time.

(3) Physics is macroscopically Lorentz invariant and isotropic but the underlying lattice is not; this is somewhat plausible since similar phenomena can be seen in certain cellular automata with regard to space isotropy.[4] However, if there is a rigid structure then curvature must result from lattice defects: This is an ungainly way to do general relativity.

(4) Space-time is discrete, but the lattice can only be strictly defined in the nonrelativistic limit; relativistically (or quantum mechanically) the lattice points are ill defined in such a way that it is not even possible to construct an average $\rho(x)$ in the flat-space state.

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[1] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Phys. Rev. Lett. **59**, 521 (1987)

[2] See, for instance, T. D. Lee, Phys. Lett. **112B**, 217 (1983).

[3] N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. **B202**, 89 (1982).

[4] For instance, lattice gases or any cellular automata based on diffusive processes such as diffusion-limited aggregation. See also N. Packard and S. Wolfram, J. Stat. Phys. **38**, 901 (1985).

