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String-Driven Inflation

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It is argued that, in fundamental string theories, as one traces the universe back in time a point is reached when the expansion rate is so fast that the rate of string creation due to quantum effects balances the dilution of the string density due to the expansion. One is therefore led into a phase of constant string density and an exponentially expanding universe. Fundamental strings therefore seem to lead naturally to inflation.

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At first sight the attractive idea that cosmic strings[1,2] or even fundamental strings[3] played a role in cosmology appears not to go very naturally with the idea of inflation.[4] Classically, during a period of exponential expansion, any string present in the universe would simply be conformally stretched, its length growing as a, the scale factor. Since the volume of the universe scales as a^3, the string density rapidly becomes negligible. If one insists on inflation, the only way to have cosmic strings play a significant role is to form the strings at the end of or after an inflationary era.[5,6,7] However, in this Letter I shall show that for infinitely thin "fundamental" strings there is a more interesting possibility. As the universe is traced back into the past, quantum effects created strings at a faster and faster rate until a point is reached where the string density approaches a constant. One is therefore automatically led back into a period of exponential expansion, i.e., inflation. Far from being incompatible with inflation, fundamental strings seem to imply it!

Independently of the present work, Aharonov, Englert, and Orloff[8] recently conjectured that such a situation might occur, where the Hawking temperature of the initial De Sitter space-time is equal to the string "limiting temperature." The calculations I report here lend support to this conjecture, although, as I shall explain, the exact numerical factors are difficult to check.

In theories based on closed strings, such as heterotic strings, there is a fundamental relation between Newton's constant G, the string tension μ, and the gauge coupling constant g: $G\mu = g^2/32\pi^2 \approx 10^{-3}$.[9] This is too large (but only just!) for these strings to exist today—one such string across the horizon would cause unacceptable distortion of the microwave background.[10] However, in the heterotic theories the fundamental strings become attached to axion domain walls at the QCD scale and thereafter rapidly disappear[3] so that there would be no such conflict with observation. In theories with both open and closed strings, such as the type-1 superstring, $G\mu$ is proportional to $(l_{\rm Pl}/R)^3$, where $l_{\rm Pl}$ is the Planck length and R is the radius of the extra six-dimensional space. There may even be models where $G\mu \approx 10^{-6}$, as required to form galaxies and clusters,[11] and where the strings do not disappear.

In any case, I will ignore the interesting issue of whether fundamental strings can form galaxies like cosmic strings and merely assume that there were fundamental closed strings with $G\mu \ll 1$ present in the very early universe. I will also assume that compactification, if necessary, has occurred and only the four-dimensional string modes may be excited. I will ignore interactions except insofar as they allow the string network to reach thermal equilibrium—the string coupling constant is in any case not known, being determined by the expectation value of the dilaton field.

Let me begin by reviewing what is known about string dynamics in an expanding universe. At low densities the strings are well out of thermal equilibrium and a network of strings evolves just as cosmic strings do.[12] As one proceeds back in time, the string density approaches a density $\rho \approx \mu^2$ and a phase transition occurs[13,14] where infinite Brownian strings and a scale-invariant distribution of loops are formed. At this point the expansion time H^{-1} is $(G\mu)^{-1/2}\mu^{-1/2} \gg \mu^{-1/2}$, which is the typical scale on the string network. It seems safe therefore to assume that the strings have reached thermal equilibrium. The fact that fundamental strings have a limiting temperature $\approx \mu^{1/2}$ plays an interesting role here—it means that the *radiation,* as long as it is in thermal contact with the string, cannot attain a density higher than $\approx \mu^2$—so as the universe contracts, it becomes *string dominated.* Note that whilst the *canonical* ensemble breaks down at these densities, the more fundamental *microcanonical* ensemble is still perfectly well defined.[13]

What happens at still higher densities? Let us begin by considering a single long straight string in an expanding background. I shall consider a De Sitter background for definiteness and calculational simplicity, but the string creation that occurs would happen in any expanding background. The equation of motion for small transverse oscillations $\mathbf{y}(x,t)$ [$\mathbf{y}$ is the *co-moving* displacement] about a long straight string along the x axis is[15]

$$\ddot{\mathbf{y}} + 2(\dot{a}/a)\dot{\mathbf{y}} = \mathbf{y}'', \tag{1}$$

where $\dot{\mathbf{y}} \equiv \partial \mathbf{y}/\partial\eta$, $\mathbf{y}' \equiv \partial\mathbf{y}/\partial x$ and I use coordinates in which the metric is conformally flat: $ds^2 = dt^2 - a^2 dx^2 = a^2(\eta)(d\eta^2 - dx^2)$ with $a = e^{Ht} = -1/H\eta$, where H is Hubble's constant and $-\infty < \eta < 0$ is conformal time. Equation (1) is exactly the same as the equation for a minimally coupled massless scalar field, which has been extensively studied in the context of inflation.[16] The solution to (1) is

$$\mathbf{y} = \frac{1}{\sqrt{\mu}} \sum_{k = n\pi/L} [a_k \boldsymbol{\chi}_k^+(\eta)\sin(kx) + a_k^* \boldsymbol{\chi}_k^-(\eta)\sin(kx)] \tag{2}$$

for a straight string of length L and with fixed end points. The canonical conjugate momentum is $\boldsymbol{\pi} = \mu a^2 \dot{\mathbf{y}}$; imposing the canonical commutation relations yields $[a_k, a_{k'}^*] = \delta_{k,k'}$, as long as the mode-function components $\chi_k(\eta)$ are normalized by the conserved norm $ia^2(\chi_k^* \dot{\chi}_k - \dot{\chi}_k^* \chi_k) = 2/L$. χ_k^+ and χ_k^- are "positive" and "negative" frequency modes. χ_k^+ is given in general by

$$\chi_k^+ = (-\eta)^{3/2} H(\pi/2L)^{1/2} [c_1 H_{3/2}^1(-k\eta) + c_2 H_{3/2}^{1*}(-k\eta)], \tag{3}$$

and $\chi_k^- \equiv \chi_k^{+*}$. Here $H_{3/2}^1(x) = (2/\pi x)^{1/2}(-1 - i/x)e^{ix}$. χ_k is correctly normalized if $c_1 c_1^* - c_2 c_2^* = 1$. The behavior of a mode χ_k is very simple. As long as the physical wavelength ak^{-1} of a mode is inside the Hubble radius, χ_k oscillates with constant *physical* amplitude $a\chi_k$. As the physical wavelength grows it crosses the Hubble radius and the *co-moving* amplitude χ_k becomes "frozen," so that the physical amplitude grows as a. Returning to (3), if we quantize the modes and define the vacuum state by $a_k|0\rangle = 0$, then different choices of vacuum correspond to different choices of c_1 and c_2. In De Sitter space the "adiabatic vacuum" or "Bunch-Davies" vacuum is defined by $c_1 = 1$, $c_2 = 0$, and this (Heisenberg) state is the state that I shall assume the string is in.[17]

In this state one calculates, for example, the mean square transverse displacement

$$\langle \mathbf{y}^2 \rangle \equiv \int_0^L \frac{dx}{L} \langle 0 | \mathbf{y}^2(x) | 0 \rangle$$

$$= \frac{DH^2\eta^2}{2\mu\pi} \int_0^\infty dk \left| \frac{1}{k} + \frac{1}{k^3\eta^2} \right|, \tag{4}$$

where the k sum has been replaced by an integral, and I include D transverse modes. The first term in (4) is the usual flat-space divergence: The physical displacement $\mathbf{y}_p \equiv a\mathbf{y}_p$ has the same divergence $(D/2\pi\mu)\int dk/k$ as in flat space. I subtract this divergence. The second term is a new divergence in curved space-time. However, if one considers it mode by mode in the context of a finite amount of exponential expansion, it is easily understood in the "adiabatic" subtraction scheme described in Ref. 17, for example. Modes with $k \gg EH$, where E is the total e-folding factor, are always within the Hubble radius and their amplitude is unaffected by the expansion. They are subtracted from (4). Modes with $k \ll H$ are always well outside the horizon and simply match on adiabatically to the modes before and after inflation. These are also subtracted. One therefore finds

$$\langle \mathbf{y}^2 \rangle \approx \frac{DH^2}{2\mu\pi} \int_H^{EH} \frac{dk}{k^3} \approx \frac{D}{4\pi\mu} \tag{5}$$

dominated by the lowest modes. One can picture this result by saying that the modes with wavelength of the order of the Hubble radius have a physical fluctuating "width" $\langle \mathbf{y}_p^2 \rangle \approx 1/\mu$ which gets amplified by the expansion after they pass out of the Hubble radius. Higher-k modes have to wait longer to cross the Hubble radius (crossing at $a = k/H$) and so they lose out in growth.

More interestingly, one can calculate the energy acquired by each mode in this process. For small $k\mathbf{y}$ the energy is given by[15]

$$e = \mu a \int dx (1 + \tfrac{1}{2}\mathbf{y}'^2 + \tfrac{1}{2}\dot{\mathbf{y}}^2), \tag{6}$$

where the first term is just the classical stretching. Now, just as in (5), we find

$$\langle \mathbf{y}'^2 \rangle \frac{DH^2}{2\mu\pi} \int_H^{EH} \frac{dk}{k} = \frac{DH^2}{2\mu\pi} \ln(E). \tag{7}$$

The $\langle \dot{\mathbf{y}}^2 \rangle$ term gives no contribution after the flat-space subtraction. Thus we deduce that the fractional energy in the perturbation grows linearly with time. This is because each mode receives a boost $k^2\mathbf{y}_k^2 \approx H^2/\mu$ on crossing the Hubble radius. $k^2\mathbf{y}_k^2$ remains constant thereafter as the wave is conformally stretched. Thus all modes contribute equally to the energy. In fact, if one cuts off the k integral for $k>\xi^{-1}$, i.e., "smoothing out" the string on a scale ξ, one finds the total length is proportional to $1+(DH^2/4\mu\pi)\ln(1/H\xi) \approx (H\xi)^{\epsilon}$, with $\epsilon = DH^2/4\mu\pi$. Writing $L=R^{\beta}/\epsilon(\beta-1)$, where R is the course-grained distance, we find that β, the fractal dimension of the string, is given by $\beta = 1+DH^2/4\mu\pi$. From this one sees that with cosmic strings of the Nielsen-Olesen type these quantum effects are usually small. This is because the width of the string is $\approx \mu^{-1/2}$ and this must be less than the Hubble radius in order for the string not to be "pulled apart" into its constituent fields by the expansion. But if $\mu^{-1/2} \ll H^{-1}$, the induced fluctuations are small and the fractal dimension close to unity.

Now the above analysis is only valid for perturbations $\mathbf{y}$ smaller than their wavelengths. But we are interested precisely in the case when this is not true—when a length of string *larger* than the length originally present is created per expansion time. The above analysis does indicate the possibility of this happening—for large enough H^2/μ we can apparently produce unlimited quantities of string per expansion time. Is this correct? For arbitrary large-amplitude motions the string equations are, in fact, not very different from (1) [15]:

$$\ddot{\mathbf{y}} + 2(\dot{a}/a)\dot{\mathbf{y}}A = (1/\epsilon)\partial_\sigma(\mathbf{y}/\epsilon). \tag{8}$$

Here $A \equiv 1-\dot{\mathbf{y}}^2$, $\epsilon^2 \equiv \partial_\sigma \mathbf{y}^2/(1-\dot{\mathbf{y}}^2)$, and σ parametrizes the length of the string. The most important term is A which couples the string to the background. Certainly for $A=0$ there would be no string creation. However, classically $\langle \dot{\mathbf{y}}^2 \rangle = \frac{1}{2}$ for excited modes well inside the horizon, and this is only reduced near the horizon, where most of the string creation occurs. So we have $\frac{1}{2} < A < 1$. In fact, reducing A to $\frac{1}{2}$ results in Hankel functions of order $\sqrt{2}$ instead of $\frac{3}{2}$ in (3), with little diminution of the string-creation effect at the Hubble-radius crossing. What about ϵ^2? For helical waves we can take ϵ to be independent of σ, and $\epsilon^{-2} = (1-\dot{\mathbf{y}}^2)/\partial_\sigma \mathbf{y}^2$. Now there are two effects which conspire to *weaken* the curvature term in (8) relative to that in (1). First, $1-\dot{\mathbf{y}}^2 < 1$ and, second, $(\partial_\sigma \mathbf{y})^2 > (\partial_\sigma x)^2$, where x is the x component of the vector $\mathbf{y}$. Both of these effects are in any case of order unity. Following through the analysis from (1) we see that weakening the $\mathbf{y}''$ term only *increases* the amplitude of the induced fluctuations. Thus the estimate of $\mathbf{y}_p^2 \approx 1/\mu$ at Hubble-radius crossing is still certainly valid.

The most serious consequence of the linearized calculation is, however, to ignore the fact that creation of string modes larger than the horizon produces more string which will in turn produce further string. We can account for this in a phenomenological equation,

$$\partial_t \rho_s = -\alpha H \rho_s + \beta(H^3/\mu\pi)\rho_s, \tag{9}$$

where the first term is the dilution of string density due to the expansion and the second is due to string creation. α and β are coefficients of order unity. For long strings such as in the calculation above, $\alpha=2$ and $\beta=D/4$, and a small energy perturbation obeys $\partial_t(\delta\rho_s a^2) = (DH^3/4\mu\pi)\rho_s a^2 = \text{const}$, so that $\delta\rho_s a^2 \propto t$ in agreement with (6) and (7). However, the full solution is, of course, exponential growth of $\delta\rho_s a^2$.

Now let us try to self-consistently feed back the effect of string creation into the expansion rate of the string-dominated universe. Assuming a flat universe (any exponential expansion would quickly make the universe very flat), we can substitute $(8\pi G/3)\rho_s$ for H^2 in (9). Now we see from (9) that there is an unstable fixed point at $H^2 = H_{\rm sdi}^2 \equiv \alpha\mu\pi/\beta$ or $\rho_s = 3\alpha\mu/8\beta G \equiv \rho_{\rm sdi}$. If the density is near $\rho_{\rm sdi}$, then it remains nearly constant as the universe expands. Thus the universe expands exponentially and quickly becomes flat and homogeneous.

Exactly how we got into the state $\rho \approx \rho_{\rm sdi}$ in the first place remains a mystery at this point—for the moment one can only say that it is a phenomenological fact that as we trace the universe back in time we are led into an exponentially expanding phase.

It is interesting to compare the above formula for $\rho_{\rm sdi}$ with that conjectured by Aharonov, Englert, and Orloff.[8] They equate the Hawking temperature $H/2\pi$ with the string limiting temperature $T_{\rm lim} = (3\mu/\pi D)^{1/2}$.[13] Thus they obtain $H^2 = 12\mu\pi/D$. In fact if we are in four dimensions, with $\alpha=3$ (because highly convoluted strings

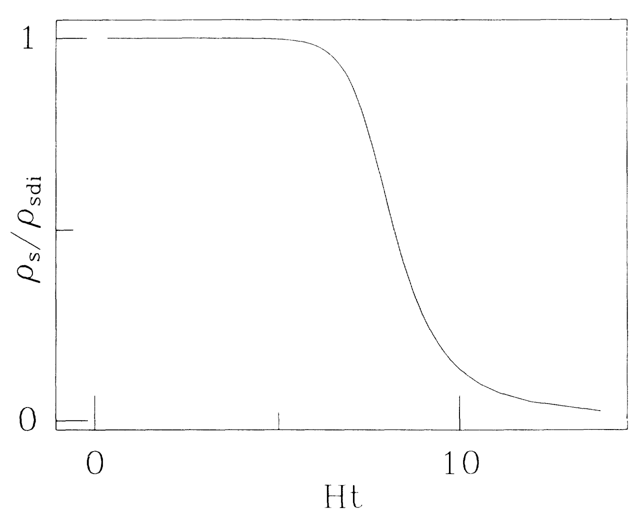

FIG. 1. As the universe is traced back in time it becomes string dominated. The curve shows the string density ρ_s as a function of time as one goes back still further. The string density asymptotically approaches a constant, $\rho_{\rm sdi} = 9\mu/4G \approx 10^{-3}$ of the Planck density for the heterotic string, for example.

behave like matter classically, $\rho_s \propto a^{-3})$[15] and $D=2$, so $\beta=\frac{1}{2}$ as above, we recover exactly the same result! Of course as I said this is only an approximation of the true result—the coincidence is intriguing nevertheless.

The trajectories of (9) are given, for $\rho < \rho_{\text{sdi}}$ by

$$H_{\text{sdi}}t = \frac{1}{\sqrt{z}} + \ln\left[\frac{1-\sqrt{z}+(1-z)^{1/2}}{1+\sqrt{z}+(1-z)^{1/2}}\right] + \text{const}, \quad (10)$$

where $H_{\text{sdi}}^2 = (8\pi G/3)\rho_{\text{sdi}}$ is the fixed-point Hubble constant and $z=\rho/\rho_{\text{sdi}}$. This is shown in Fig. 1. Clearly if we start with $\rho=\rho_{\text{sdi}}(1-\delta)$, $\delta \ll 1$, the density remains approximately constant for $\approx \frac{1}{2}\ln(1/\delta)$ Hubble times. To obtain enough inflation, for example, to "solve" the horizon problem we require an initial value of $\delta \approx e^{-100} \approx 10^{-40}$.

This seems very small—if one assumes, for example, that δ is Gaussian distributed about zero with dispersion $\sigma \approx 1/\sqrt{N}$, where $N \approx 1/G\mu$ is the number of long strings per Hubble volume, then the fraction of space where δ is so small would be tiny. However, the volume where δ is small gets inflated by $\exp(3H_{\text{sdi}}t_I)$, with the e-folding factor $H_{\text{sdi}}t_I = \frac{1}{2}\ln(1/\delta)$. Thus the fraction of the *present* universe occupied by regions where δ was between δ and $\delta+d\delta$ is proportional, for $\delta \ll \sigma$, to $\delta^{-3/2}d\delta$. Thus most of the universe would still be inflating! From this viewpoint, such a small initial value of δ in a region of the universe as old as ours would be very likely indeed.

Let me summarize the findings of this paper. If we follow our observable universe back in time into the very early universe, at a density $\rho \approx (G\mu)^2\rho_{\text{Pl}}$, where ρ_{Pl} is the Planck density, a phase transition occurs and the universe becomes dominated by very long strings. As we proceed back to higher densities we approach $\rho_{\text{sdi}} \approx (G\mu)\rho_{\text{Pl}}$ where the Hubble radius is $\approx \mu^{-1/2}$. The universe is expanding exponentially and in consequence has become very flat and homogeneous. At this point the Hawking temperature of the DeSitter space is of the same order as the string limiting temperature. The mean separation of the strings is $\approx l_{\text{Pl}}$, the Planck length, and one might expect that string interactions prevent the density from growing any higher.

The calculations reported here are very preliminary and certainly leave many questions unanswered. How large are the fluctuations in the initial DeSitter spacetime; does this scenario have the same "fluctuation problem" that most inflation scenarios do? What are the initial conditions for the universe (or perhaps just for our region of the universe), and how long does the exponentially expanding phase last? It is interesting to note that in a collapsing region of the universe, as argued above, the *radiation* density is limited by the presence of long strings—but is the string density itself limited, perhaps by string-string interactions? If so, what happens to the trajectories for which $\rho > \rho_{\text{sdi}}$? Lastly, it would be very interesting to try and describe the "string-driven inflation" state in terms of string field theory, perhaps along the lines of the work of Horowitz *et al.*[18]

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