Mechanism for Superuniversal Behavior in Certain Stochastic Systems

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The problems of a directed polymer in a random matrix, a randomly stirred fluid obeying Burgers's equation, and the dynamics of an interface growing by vapor deposition can be mapped into each other. Renormalization-group arguments suggest that there is an upper critical dimension for these systems above which the correlation-length exponent ν should be $\frac{1}{2}$. We present an argument which suggests that $n = \frac{2}{3}$ in all dimensions (that is, it is superuniversal), in agreement with the conjecture and numerical simulations of Kardar and Zhang.

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The seemingly disparate problems of a directed walk on a random lattice,[1] the growth of an interface due to vapor deposition or following an Eden-model growth law,[2] and a randomly stirred fluid whose hydrodynamics is approximated by Burgers's equation[2] are all mathematically equivalent to each other and examples of systems for which the conventional renormalization-group (RG) approach fails completely.[1] Because of the mappings between these problems it is only necessary to outline our calculations in detail for just one of them, which we choose to be that of the directed walk on a random lattice.

A directed walk on a lattice is one in which no reverse steps are ever taken along a particular direction, called the longitudinal direction. Suppose the directed walk starts at the origin and ends after t steps have been taken along the longitudinal direction. Then a typical displacement $|\mathbf{x}|$, where $\mathbf{x}$ is a $(d-1)$-dimensional vector in the transverse directions, scales as t^{ν}, with $\nu = \frac{1}{2}$, for all dimensions d, as would be expected for a random walk. The walk in the presence of quenched random disorder, such as might be provided by some missing lattice bonds, can be mapped in any dimension onto the randomly stirred Burgers equation of hydrodynamics. For $d=2$, this is exactly solvable[3,4] and gives $\nu = \frac{2}{3}$. Conventional RG calculations[1,3] suggest that there is the upper critical dimension in the presence of disorder and that $\nu = \frac{1}{2}$ for all $d > 3$. However, numerical transfer-matrix calculations by Kardar and Zhang[1] (KZ) gave $\nu = 0.62 \pm 0.04$ for $d=3$ and $\nu = 0.64 \pm 0.07$ for $d=4$, suggesting that the value $\frac{2}{3}$ for ν might be superuniversal, i.e., the same for all dimensions. Here we introduce a mechanism which shows how superuniversal exponents could arise, and using a simple analytic argument based on replica methods, we obtain the superuniversal value of $\frac{2}{3}$ for ν.

The problem of a directed walk on a random lattice is related to that of a directed polymer (possibly a polyelectrolyte according to KZ) in a random matrix. (Self-intersections are unlikely and irrelevant for a directed walk because of the "stretching" along the longitudinal direction; a self-avoiding directed walk in the absence of a disorder has $\nu = \frac{1}{2}$ for all d.[5]) The transverse displacements in the presence of quenched disorder are related to the conformations of the polyelectrolyte in a frozen gel matrix.

The partition function $Z(\mathbf{x},t)$ of all directed polymers joining $(\mathbf{0},0)$ and $(\mathbf{x},t)$ can be expressed as the path integral[1]

$$Z(\mathbf{x},t) = \int_{(\mathbf{0},0)}^{(\mathbf{x},t)} D^3x'(t') \exp\left\{ -\int_0^t dt' \left[\frac{\gamma}{2} \left(\frac{d\mathbf{x}'}{dt'} \right)^2 + \mu(\mathbf{x}',t') \right] \right\}. \tag{1}$$

The continuum limit has been taken in Eq. (1). The term involving γ is an "entropic" contribution arising from the chain connectivity while the term in $\mu(\mathbf{x}',t')$ arises as different conformations of the chain which pass through different "impurities" have differing energies. Following KZ, we shall assume that

$$\langle \mu(\mathbf{x},t)\mu(\mathbf{x}',t') \rangle_c = \delta(t-t') V(\mathbf{x}-\mathbf{x}'), \tag{2}$$

where the c denotes a configuration average over impurity positions, etc. All $V(\mathbf{R})$ which decay to zero on the cutoff (lattice spacing) length scale would be expected to be in the same universality class. KZ set $V(\mathbf{R}) = \sigma^2 \delta^{d-1}(\mathbf{R})$. An alternative to performing the path integral of Eq. (1) is to solve the non-Markovian diffusion equation

$$\partial Z/\partial t = [(\gamma/2)\nabla^2 + \mu(\mathbf{x},t)] Z. \tag{3}$$

When $\mu(\mathbf{x},t) = 0$, i.e., without the disorder,

$$Z(\mathbf{x},t) \sim t^{-(d-1)/2} \exp(-x^2/2\gamma t), \tag{4}$$

which implies $\langle x^2 \rangle = \gamma t$. Hence $\nu = \frac{1}{2}$ in the absence of impurities. In the presence of a quenched distribution of impurities one expects that for a typical impurity distribution

$$Z(\mathbf{x},t) \sim t^{-(d-1)\nu} g(x/t^{\nu}) t^{\gamma-1} \exp(-ft), \tag{5}$$

where f is the polymer free energy per step in the longitudinal direction and γ is here the "susceptibility" exponent.[6] In order to calculate f we must average the logarithm of $Z(\mathbf{x},t)$ over the impurities (rather than $Z(\mathbf{x},t)$ itself) so as to obtain results representative of a typical impurity distribution.

The nonlinear transformation $Z(\mathbf{x},t) = \exp[\lambda h(\mathbf{x},t)/\gamma]$ gives the equation

$$\partial h/\partial t = \tfrac{1}{2}\gamma \nabla^2 h + \tfrac{1}{2}\lambda(\nabla h)^2 + (\gamma/\lambda)\mu(\mathbf{x},t). \tag{6}$$

This describes the fluctuations of a growing interface in a ballistic deposition process.[2] Setting $\mathbf{v} = -\lambda \nabla h$, one has the Burgers equation

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \tfrac{1}{2}\gamma \nabla^2 \mathbf{v} - \gamma \nabla \mu(\mathbf{x},t) \tag{7}$$

for a randomly stirred vorticity-free ($\nabla \times \mathbf{v} = 0$) fluid.[3]

Forster, Nelson, and Stephen[3] performed a perturbative dynamic RG calculation starting from Eq. (7) for the effective coupling constant $g = 2\sigma^2/\gamma$ [where $\sigma^2 = \langle \mu(\mathbf{x},t)^2 \rangle_c$] which gave to one-loop order the equation[1]

$$dg/dl = (3-d)g + C_{d-1}K_{d-1}g^2, \tag{8}$$

where $K_d = S_d/(2\pi)^d$ and $C_d > 0$. For $d < 3$ the only fixed point of Eq. (8) is the Gaussian fixed point $g^* = 0$, which is unstable. This implies that disorder is relevant and that flow is towards a strong-coupling regime. For $d > 3$ the Gaussian fixed point is stable, which would normally be taken as an implication that disorder is irrelevant and hence that $\nu = \frac{1}{2}$, $d > 3$. It is noteworthy that Eq. (8) has an unstable fixed point when $d > 3$, which could imply the existence of a strong-coupling regime for $g > g^* (\neq 0)$. The conventional RG discussion seems unable to describe the strong-coupling regime.

However, progress is possible if one uses the replica method to calculate first the average of $\ln Z$,

$$\langle \ln Z \rangle_c = \lim_{n\to 0} [\langle Z^n \rangle_c - 1]/n. \tag{9}$$

n copies of the polymer are generated and the average performed over the $\mu(\mathbf{x},t)$. One finds

$$\langle Z^n(\mathbf{x},t)\rangle_c = \int_{(\mathbf{0},0)}^{(\mathbf{x},t)} \prod_{\alpha=1}^{n} D^3 x'_\alpha(t') \exp\left\{ -\int_0^t dt' \left[\frac{\gamma}{2} \sum_\alpha \left(\frac{dx'_\alpha}{dt'} \right)^2 - \sum_{\alpha<\beta} V(\mathbf{x}'_\alpha(t') - \mathbf{x}'_\beta(t')) \right] \right\}. \tag{10}$$

Evaluation of this path integral is equivalent to our solving the (d-1)-dimensional n-particle Schrödinger equation with Hamiltonian

$$H_n = -\frac{\gamma}{2} \sum_\alpha \frac{\partial^2}{\partial \mathbf{x}_\alpha^2} - \sum_{\alpha<\beta} V(\mathbf{x}_\alpha - \mathbf{x}_\beta). \tag{11}$$

The potential-energy term represents an attractive interaction between the n particles. The free energy f of Eq. (5) is related to the ground-state energy $E_0(n)$ of H_n by $f = E_0(n)/n$, $n \to 0$.

In general, the n-body problem posed by Eq. (11) is intractable. However, for the case when $d = 2$ and $V(R) = \sigma^2 \delta(R)$, it is possible to calculate the ground-state wave function and energy exactly.[7] For $n = 2$, the ground-state wave function is $\sim \exp - \kappa |x_1 - x_2|$. The discontinuity in the wave function as the two particles cross is matched to the strength of the attractive δ-function interaction and fixes $2\gamma\kappa = \sigma^2$. This generalizes in the n-body case to

$$\Psi_0 \sim \exp\left(-\kappa \sum_{\alpha<\beta} |x_\alpha - x_\beta| \right), \tag{12}$$

and gives for the energy of this n-body bound state[7]

$$E_0(n) = -(\gamma/6)\kappa^2 n(n^2-1), \tag{13}$$

and so $f = \gamma\kappa^2/6$. When $d \neq 2$, we cannot find the ground-state wave function $\Psi_0(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ or the energy analytically. However, we will now argue that for n large and $V(\mathbf{R}) > 0$, all $\mathbf{R}$, the ground state is an n-body bound state, even for potentials which do not produce a two-particle bound state.

We begin by constructing a trial wave function of the form

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \prod_{\alpha=1}^{n} \phi(\mathbf{x}_\alpha - \mathbf{X}),$$

where $\mathbf{X} = (1/n)(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n)$ is the center-of-mass coordinate. Taking the ϕ to be normalized, we see that the kinetic-energy term in $\langle \Psi | H_n | \Psi \rangle$ consists of n equal contributions each of which is the kinetic energy associated with the one-particle wave function. Similarly, the potential energy consists of $\frac{1}{2}n(n-1)$ equal contributions. Suppose for concreteness we take $V(\mathbf{R})$ to be a potential well of linear dimension a and strength $\sigma^2 a^{-d}$. Then we can choose the trial wave function so that the potential-energy contribution is of the order of $-\sigma^2 n(n-1)/a^d$. Clearly, for large enough n the potential energy will dominate and the ground state will be an n-body bound state. We shall therefore proceed on the assumption that one should determine the ground-state energy for n large and analytically continue that expression for $E_0(n)$ to $n = 0$. Of course, it may be that for some smaller value of n a ground state of, for instance, $(n-m)$- and m-body bound states is favored. Therefore, our assumption is equivalent to the assumption that replica symmetry holds.

To determine the exponent ν it is convenient to add to H_n a term $-c\sum_{\alpha=1}^{n} \delta(x_\alpha^1)$: x_α^1 denotes the first component of $\mathbf{x}_\alpha$. Physically such a term corresponds to a layer of attractive impurities in the $x^1=0$ plane which tend to pin the polymer in that plane. It will localize the n-body bound state in the vicinity of the hyperplane $X \equiv \sum_\alpha x_\alpha^1/n=0$. Such an attractive interaction adds a singular term of the form $c^{1/(1-\nu)}$ to the polymer free energy $f(c)$, i.e.,

$$f(c)-f(0) \sim -c^{1/(1-\nu)}, \quad c>0, \quad c \to 0. \tag{14}$$

Hence if we can determine $f(c)$ we can evaluate ν.

In order to see how the relation (14) comes about, let us first recall one of the well-known relationships between the statistics of polymers on a lattice and magnetic critical points. Suppose we form the generating function $\chi(T)$ by running over all the directed walks of t steps on the lattice via

$$\chi(T) = \sum_t \left[\sum_{\mathbf{x}} Z(\mathbf{x},t) \right] (J/k_{\rm B}T)^t. \tag{15}$$

This generating function is analogous to the susceptibility of the magnetic system. It will diverge at the critical temperature T_c, which, with use of Eq. (5), is given by $\ln(J/k_{\rm B}T_c)=f$. We shall discuss below how the addition of the extra term to the Hamiltonian shifts T_c in the magnetic system according to $T_c(c)-T_c(0) \propto c^{1/(1-\nu)}$ for small $c>0$. This, together with the identification of f, leads to (14).

The formula for the shift in T_c, for small c, has been proved rigorously by Diehl, Dietrich, and Eisenriegler[8]; however, the result may be obtained by the following heuristic argument.[9,10] Consider a fluctuation which tends to order the system near the hyperplane $X=0$. The spatial extent of the fluctuation is governed by the correlation length ξ and it will lead to an increase in the bulk free energy by an amount $\epsilon^{2-\alpha}\xi$, where ϵ is the reduced temperature. However, for $c>0$, the extra term in the Hamiltonian, which should behave like a slightly altered layer of exchange bonds in the magnetic system analog, acts to reduce the free energy by an approximate amount $c\epsilon^{1-\alpha}$, for small c. Therefore, the total change in the free energy due to the fluctuation is

$$\delta F \sim \epsilon^{2-\alpha}\xi - c\epsilon^{1-\alpha} \sim \epsilon^{1-\alpha}(\epsilon^{1-\nu}-c). \tag{16}$$

Thus the critical temperature at which the fluctuation is energetically favored is given by $\epsilon_c^{1-\nu} \sim c$ or $T_c(c) - T_c(0) \sim c^{1/(1-\nu)}$.

The replicated Hamiltonian, including the modified-layer contribution, can be written as a sum of two terms, H_n^0+V, where

$$H_n^0 = H_n - nc\delta(X), \tag{17}$$

$$V = -c\left[\sum_{\alpha=1}^{n} \delta(x_\alpha^1) - n\delta(X) \right]. \tag{18}$$

The center-of-mass motion in the X direction can be separated out from H_n^0 to yield

$$H_{\rm c.m.} = -(\gamma/2n)\partial^2/\partial X^2) - nc\delta(X). \tag{19}$$

The lowest eigenstate of this Hamiltonian is a bound state of energy $-n^3c^2/2\gamma$ and eigenfunction $\sqrt{\lambda} \times \exp(-\lambda|X|)$, where $\lambda = n^2c/\gamma$. Thus the ground-state wave function of H_n^0,

$$|0\rangle = \sqrt{\lambda}\exp(-\lambda|X|)\Psi_0(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n), \tag{20}$$

has energy

$$E_n^0(c) = E_0(n) - n^3c^2/2\gamma. \tag{21}$$

The "perturbation" V gives contributions to the ground-state energy only of order c^3. To first order we have

$$\langle 0|V|0\rangle = \lambda \int \prod_\alpha d^3x_\alpha [1-2\lambda|X|+\cdots]\Psi_0^2(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)V, \tag{22}$$

where the exponential in $|0\rangle$ has been expanded out to order c. The terms in Eq. (22) of order c^2 cancel. This is a consequence of the translational invariance of $\Psi_0(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$; adding a constant vector $\mathbf{a}$ to all $\mathbf{x}_\alpha$ leaves Ψ_0 unchanged. The leading term is then of order c^3: viz.,

$$2(n^4c^3/\gamma^2)\int \prod_\alpha d^3x_\alpha |X| \Psi_0^2(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\sum_\alpha \delta(\mathbf{x}_\alpha^1).$$

The second-order term in V, $-\sum_m \langle 0|V|m\rangle^2/[E_m^0 - E_n^0(c)]$, where $|m\rangle$ and E_m^0 denote the mth excited eigenstate and eigenvalue of H_n^0, also gives contributions of order c^3 to the ground-state energy. Each matrix element $\langle 0|V|m\rangle$ is, at most, of order $c^{3/2}$. The sum over m is convergent even when $E_m^0 - E_n^0(c)$ is replaced by its $c=0$ limit. These statements have been verified explicitly for the "solvable" case of $d=2$. Third-order terms in V give only contributions higher than c^3. Thus if $E_0(n,c)$ denotes the ground-state energy of the full Hamiltonian H_n^0+V,

$$E_0(n,c) = E_0(n) - n^3c^2/2\gamma + O(c^3), \tag{23}$$

which implies that $f(c)-f(0) \approx O(c^3)$ and so, with Eq. (14), that $\nu = \frac{2}{3}$. Notice that for $c<0$, that is a repulsive surface layer of impurities, no surface bound state would form and $E_0(n,c) = E_0(n)$ for all $c<0$.

The essential features of our calculation are (i) the vanishing of the contribution of the c^2 term in $f(c)$ because of the n^3 dependence of the c^2 term in $E_0(n,c)$ and (ii) the assumed existence of a c^3 term in $E_0(n,c)$ of the form $nc^3D(n,c)$ where $D(0,c)$ is nonzero and finite as $c \to 0$. The first feature should obtain for any dimension provided that replica symmetry holds and implies that $\nu \neq \frac{1}{2}$. The second feature is crucial for the prediction of the superuniversal value for ν of $\frac{2}{3}$. $D(n,c)$ presumably has these features for $d=2$, since with these assumptions one recovers the known exact results. An exact calculation of it is very difficult in general dimensionalities, as this requires complete knowledge of $|m\rangle$ and E_m^0. From an inspection of our perturbative expressions, we expect that $D(n,0)$ is nonzero and finite. However, it does remain possible that the perturbation expansion breaks down in the $n=0$ limit. For instance it could be that $D(0,c) \sim c^{-\rho(d)}$ for $d \neq 2$, which would imply a breakdown of superuniversality [but not necessarily $\nu = \frac{1}{2}$ unless $\rho(d)=1$]. Such a possibility might explain the very recent numerical simulations of Wolf and Kertész,[11] who claim that $\nu(d=3)=0.60$ and $\nu(d=4)=0.57$. However, the approach to the strong-coupling fixed point away from the unstable fixed point may be slow, making the numerical determination of ν difficult. Should it finally emerge that ν is not really superuniversal, then the calculation of $D(0,c)$ offers a means of determining ν. Within the field-theoretic approach of Ref. 3, there is no obvious way of even starting such a calculation for $d \geq 3$.

It has been suggested by KZ that, at least for $d>3$, there exist two regimes—a strong-coupling regime with $\nu = \frac{2}{3}$ and a weak-coupling regime where $\nu = \frac{1}{2}$. Our calculations make this a very plausible scenario. Suppose for $d>3$ that $V(\mathbf{R})$ is indeed the square well of depth $-\sigma^2/a^d$. Then for $n=2$, unless σ^2 exceeds a critical value σ_c^2, no bound state exists. The form of the ground-state energy, $E_0(n,c)$, as $n \to 0$ would hardly be expected to follow the large-n result in this case. Indeed, if the continuation followed a branch characterized by the nonexistence of a bound state, we would expect a term nc^2 in $E_0(n,c)$ arising from the interaction of each of the n nonbound particles with the plane, which would imply $\nu = \frac{1}{2}$. On the other hand, should $\sigma^2 > \sigma_c^2$, a smooth extrapolation to $n=0$ could exist. It is less clear how universal these arguments are: Do all reasonable potentials give rise to a critical coupling σ_c^2? Further insights on this problem would be most useful as they would probably have relevance in other contexts.

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