Virasoro Algebras with Central Charge $c > 1$

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(Received 30 September 1987)

We provide evidence for a new unitary series of conformal field theories, labeled by integers M and N. For $N=1,2$ they reproduce the unitary conformal and superconformal series of minimal models. For higher N, they correspond to models with $c>1$, generated by new nonlocal currents of spin $(N+4)/(N+2)$. We use a generalization of the Feigin-Fuchs construction to find the currents and the primary fields of the new algebras.

PACS numbers 11.10.−z, 02.20.+b, 05.50.+q, 11.17.+y

The complete classification of all two-dimensional conformal field theories is a major goal in the study of critical phenomena. This is because the representations of conformal algebras play a central role in describing the critical behavior of two-dimensional statistical systems.[1,2] Conformal field theories are also important for string theory.[2,3] In string theory, each modular-invariant conformal field theory describes a possible string compactification. One hopes that the study of conformal field theories will lead to a deeper understanding of the nonperturbative aspects of strings.

In two dimensions, the conformal group is infinite dimensional. The algebra consists of two copies of the Virasoro algebra, labeled by their central charge c. Many conformal field theories are known to exist. For example, there is an infinite set of exactly solvable unitary theories with central charge

$$c=1-6/(M+2)(M+3), \qquad (1)$$

for $M \geq 1$. This set of minimal models has been extensively analyzed, and their mathematical structure is well understood.[2,4,5] The first models in this series correspond to the Ising model ($M=1$), the tricritical Ising model ($M=2$), and the three-state Potts model ($M=3$).

In this Letter, we provide evidence for new unitary series of conformal field theories. Our models are labeled by integers M and N. For $N=1$, they reproduce the unitary minimal series (1). For $N=2$, they describe the unitary superconformal models with $c<\frac{3}{2}$.[3,4] For higher N, we conjecture that our theories form unitary representations of extended Virasora algebras generated by new nonlocal currents of spin $(N+4)/(N+2)$. In what follows we use a generalization of the Feigin-Fuchs construction[6] to find the currents and the primary fields of the new algebras. The new currents are generalizations of the usual superconformal current to the case $N>2$.

The Virasora algebras associated with our models are obtained from the Goddard-Kent-Olive (GKO) construction[5] for cosets of affine Kac-Moody symmetries. The GKO construction is based on the fact that every Kac-Moody algebra g gives rise to an associated Virasoro algebra with generators L_n^g and central charge c^g. Given an algebra g and a subalgebra h, GKO showed that the operators $K_n=L_n^g-L_n^h$ generate a new Virasoro algebra with central charge $c=c^g-c^h$. Their construction holds for any algebra g and subalgebra h.

In this Letter we consider the case where $g=\mathrm{su}(2)\oplus\mathrm{su}(2)$, and h is the diagonal su(2) subalgebra. We take a level of M representation for the first factor, a level N representation for the second, and a level $M+N$ representation for the diagonal subalgebra, and use the GKO construction to build a Virasoro algebra with central charge

$$c=\frac{3MN(M+N+4)}{(M+N+2)(M+2)(N+2)}. \qquad (2)$$

Here we rely on the fact that $c^g=3k/(k+2)$ for an su(2) representation of level k. Note that (2) is manifestly symmetric under the interchange $M\leftrightarrow N$. For $N=1$, it reduces to the unitary minimal series of Eq. (1). For $M,N>1$, it describes a new series of conformal field theories with $c>1$.

The unitary minimal models (corresponding to $N=1$) can be represented in terms of a single free scalar ϕ, with stress-energy tensor[7] $T_{zz}=-\frac{1}{4}\,\partial_z\phi\,\partial_z\phi+i\alpha_0\partial_z^2\phi$. The central charge for this system is $c=1-24\alpha_0^2$. The α_0-dependent term in the stress tensor is needed to give a central charge that is less than 1. Physically, it corresponds to a background charge $-2\alpha_0$ located at infinity on the z plane. With a background charge $2\alpha_0=1[(M+2)(M+3)]^{-1/2}$, this construction reproduces the central charges of the unitary minimal models.

The primary fields of the minimal models are represented by vertex operators $V_\alpha=e^{i\alpha\phi}$. These vertex operators have conformal dimension $\Delta_\alpha=\alpha^2-2\alpha\alpha_0$, and so V_α and $V_{2\alpha_0-\alpha}$ have the same dimension and represent the same physical state. In this representation, the vertex operators of dimension 1 play a special role. They are known as Feigin-Fuchs screening operators[6,7] and have the form $V_{\alpha_\pm}=\exp(i\alpha_\pm\phi)$, where

$$\alpha_+=\frac{M+3}{[(M+2)(M+3)]^{1/2}},\qquad \alpha_-=-\frac{M+2}{[(M+2)(M+3)]^{1/2}}. \tag{3}$$

The screening operators are necessary for the correlator $\langle V_\alpha V_\alpha V_\alpha V_{2\alpha_0-\alpha}\rangle$ to be nonvanishing. Since the screening charges have conformal dimension 0, they do not change the conformal properties of the correlator.

The requirement that vertex operators have nonzero four-point functions restricts the allowed values of α. The physical vertex operators are of the form $V_{pq}=\exp(i\alpha_{pq}\phi)$, where $\alpha_{pq}=\frac{1}{2}(1-p)\alpha_++\frac{1}{2}(1-q)\alpha_-$. The V_{pq} have dimension

$$h_{pq}=\frac{[p(M+3)-q(M+2)]^2-1}{4(M+2)(M+3)}; \tag{4}$$

they are the primary fields of the Virasoro algebra for the unitary minimal models.

For general N, the central charge (2) can be written as

$$c=1-\frac{6N}{(M+2)(M+N+2)}+\frac{2(N-1)}{N+2}. \tag{5}$$

The first two terms in the above expression are the central charge for a bosonic field with a given background charge. The third term is the central charge for a Z_N parafermionic theory.[8] (Of course, the above expression could also have been written in terms of Z_M parafermions and a different background charge. This duality will play an important role later in this Letter.)

The Z_N parafermionic theories are generated by nonlocal currents of fractional spin. They can be described in terms of a free scalar field $X(z)$ and an su(2) level-N Kac-Moody algebra. The parafermionic fields Φ_m^l have dimensions

$$\Delta_m^l=l(l+2)/4(N+2)-m^2/4N,$$

where $l-m=0$ (mod 2).

The fields Φ_{N-2k}^N, for $k=1,\ldots,N-1$, are the parafermionic currents ψ_k. The dimensions of these fields are $\Delta_k=k(N-k)/N$. The dimensions of the parafermions satisfy the condition $\Delta_{N-k}=\Delta_k$, and so $\psi_k^\dagger=\psi_{N-k}$. The fields Φ_k^k, $k=1,\ldots,N-1$, are the primary fields of the parafermionic current algebra. In statistical mechanics, they correspond to spin fields σ_k, of dimension $\Delta_k=k(N-k)/2N(N+2)$. The parafermionic algebra also includes a set of Hermitean fields $\epsilon_j=\Phi_0^{2j}$, for $j=1,\ldots,[N/2]$. The fields ϵ_j have dimension $\Delta_j=j(j+1)/(N+2)$. In statistical mechanics, they represent energy operators.

The states created by the fields ϵ_j can be represented in terms of the creation operators of the parafermionic current ψ_1 acting on the highest-weight states $|\sigma_k\rangle$,

$$|\epsilon_j\rangle=\prod_{l=1}^{j}A_{(1-2l)/N}|\sigma_{N-2j}\rangle. \tag{6}$$

There are other Hermitean fields whose dimensions differ by integers from the fields ϵ_j. These fields are constructed by replacing $A_{k/N}$ in (6) by $A_{k/N-n}$, for $n\in Z_+$. Later we will use the field $\hat{\epsilon}_1$, associated with the state

$$|\hat{\epsilon}_1\rangle=A_{-1/N-1}|\sigma_{N-2}\rangle. \tag{7}$$

The field $\hat{\epsilon}_1$ has conformal dimension $(N+4)/(N+2)$.

We will now generalize the Feigin-Fuchs construction to our extended theories with $N\geq 2$. We build the screening operators from the Z_N parafermions ψ_1 and $\psi_1^\dagger$ and from bosonic vertex operators $\exp(i\alpha_\pm\phi)$. The screening operators are $V_{\alpha_+}=\psi_1\exp(i\alpha_+\phi)$ and $V_{\alpha_-}=\psi_1^\dagger\exp(i\alpha_-\phi)$, where

$$\alpha_+=\frac{M+N+2}{[N(M+2)(M+N+2)]^{1/2}},\qquad \alpha_-=-\frac{M+2}{[N(M+2)(M+N+2)]^{1/2}}. \tag{8}$$

The operators V_{α_+} and V_{α_-} have dimension 1 since the parafermions have dimension $(N-1)/N$.

All of the fields in our new theories can be constructed from a bosonic vertex operator and a field in the parafermionic theory. There is a very special set of fields, however, that we believe to be the primary fields of a new current algebra. These fields have the form $\Psi_{pq}=\sigma_k\exp(i\alpha_{pq}\phi)$, where $k=|p-q\ (\mathrm{mod}\ N)|$ and α_{pq} is as above. The primary fields Ψ_{pq} have dimension

$$H_{pq}=\frac{[p(M+N+2)-q(M+2)]^2-N^2}{4N(M+2)(M+N+2)}+\frac{k(N-k)}{2N(N+2)}, \tag{9}$$

for $k=|p-q\ (\mathrm{mod}\ N)|$. The values of p and q are restricted to the range $1\leq p\leq M+1$ and $1\leq q\leq M+N+1$. Later we shall see that the dimensions (9) can also be obtained through the characters of the su(2) algebra.

For $N=2$, this construction reproduces the unitary series of $c<\frac{3}{2}$ superconformal models.[9] The primary fields Ψ_{pq} give rise to unitary irreducible representations of the superconformal algebra that typically are reducible with respect to the Virasoro algebra. For example, the case $N=2$, $M=1$ describes the tricritical Ising model.[10] This model contains a superconformal primary field Ψ_{13} of dimension $\frac{1}{10}$. The representation space of this field splits into two irreducible representations of the ordinary Virasoro algebra, $(\frac{1}{10})_{\mathrm{sconf}}=(\frac{1}{10})_{\mathrm{Vir}}\oplus(\frac{3}{5})_{\mathrm{Vir}}$.

For $N=2$ and any value of M, the superconformal algebra is generated by the current $J_z=\partial_z\phi\psi-4i\alpha_0\partial_z\psi$ of dimension $\frac{3}{2}$. The null states of the superconformal algebra are constructed with the help of the screening operators.[11] The current J_z and the screening operators $V_{\alpha_\pm}$ have operator products of the form

$$J_z(z)V_{\alpha_\pm}(w)=-\frac{i}{\alpha\pm}\left[\frac{V_{\alpha_\pm}(w)}{(z-w)^2}+\frac{\partial_w V_{\alpha_\pm}(w)}{z-w}\right]+\cdots. \tag{10}$$

This ensures that the current commutes with the screening operators, as required for the null-state construction.

For larger values of N, the dimension of the current is determined by our setting $M=1$. This gives a "dual" description of the $c<1$ unitary models in terms of Z_N parafermions. The new current J_z always has dimension $h_{31}=(N+4)/(N+2)$. For $N=2$, the current has dimension $\frac{3}{2}$, while for $N=3$, the dimension of the current is $\frac{7}{5}$. The case $N=4$, with a nonlocal current of dimension $\frac{4}{3}$, has been investigated by Fateev and Zamolodchikov.[12] Below we shall see how the current acts in the minimal model $N=3$, $M=1$.

For $N>2$ there are two conformal fields of dimension h_{31}. They are $J_z^1=\partial_z\phi\epsilon_1-i(N+2)\alpha_0\partial_z\epsilon_1$ and $J_z^2=\tilde{\epsilon}_1$. Here ϵ_1 and $\tilde{\epsilon}_1$ are Hermitean fields defined in (6) and (7). Their operator products with the parafermion ψ_1 can be determined from the operator products of the su(2) algebra,

$$\begin{aligned}&\epsilon_1(z)\psi_1(w)=\sigma_2(w)/(z-w)+\cdots,\\&\tilde{\epsilon}_1(z)\psi_1(w)=\sigma_2(w)/(z-w)^2+\cdots.\end{aligned} \tag{11}$$

The current J_z must be a linear combination of the conformal fields J_z^1 and J_z^2. The relative coefficient is fixed by requiring the operator product of J_z and $V_{\alpha_\pm}$ to have a form similar to (10). It is not hard to show that this determines the current to be

$$J_z=\partial_z\phi\epsilon_1-i(N+2)\alpha_0\partial_z\epsilon_1+\tfrac{1}{2}i(N-2)(\alpha_+-\alpha_-)\tilde{\epsilon}_1. \tag{12}$$

For $N=2$, Eq. (12) reduces to the usual supercurrent because the energy operator ϵ_1 becomes degenerate with the parafermion ψ.

For $N=3$ and $M=1$ the above construction gives an alternative description of the $c=\frac{4}{5}$ minimal model, in terms of Z_3 parafermions and a scalar with a background charge. The primary fields of the extended algebra have dimension 0, $\frac{1}{8}$, $\frac{1}{40}$, $\frac{1}{15}$, and $\frac{2}{5}$. All the other fields in the minimal model can be obtained from these fields by applying the current J_z. The primary fields split into charge sectors according to the charge of the spin field σ_k. The neutral sector includes the states $|0\rangle$ and $|\frac{1}{8}\rangle$. The current acting on these states gives rise to $|\frac{7}{5}\rangle$ and $|\frac{21}{40}\rangle$, respectively. This follows from the operator product expansions of ϵ_1 and $\tilde{\epsilon}_1$ with the identity. The current in this sector is moded by $J_{-2/5+n}$. Similarly, the charge ± 1 sector contains the states $|\frac{1}{40}\rangle$, $|\frac{1}{15}\rangle$, and $|\frac{2}{5}\rangle$. The current acting on these states gives $|\frac{13}{8}\rangle$, $|\frac{2}{3}\rangle$, and $|3\rangle$. This follows from the operator product expansions of ϵ_1 and $\tilde{\epsilon}_1$ with the spin operators σ_1 and σ_2. In this sector, the current is moded by $J_{-3/5+n}$. Details of this construction, and the generalization to other models, will be presented elsewhere.[13]

Further evidence that the fields Ψ_{pq} are the primary fields of a new current algebra is provided by the su(2) characters. We will use these characters to decompose representations of $g=\mathrm{su}(2)\oplus\mathrm{su}(2)$ with respect to $h\oplus V$, where $h=\mathrm{su}(2)$ and V denotes the symmetry algebra of our new models. The su(2) characters $\chi_{N,l}(z,\theta)$ are labeled by the level N and spin l of the representation; they are defined in Ref. 5.

For the cases $N=1$ and 2, the algebras V are just the Virasoro and super-Virasoro algebras. Their characters can be found from the su(2) branching functions, as shown in Ref. 5. Here we generalize this construction to higher N, and define χ_{pq} as

$$\chi_{M,(p-1)/2}\sum_{m=0}^{[N/2]}\chi_{N,m}=\sum_{q=1}^{M+N+1}\chi_{M+N,(q-1)/2}\chi_{pq}, \tag{13}$$

for $p-q$ even. For $p-q$ odd, the χ_{pq} are obtained from the product

$$\chi_{M,(p-1)/2}\sum_{m=0}^{[(N-1)/2]}\chi_{N,m+1/2}=\sum_{q=1}^{M+N+1}\chi_{M+N,(q-1)/2}\chi_{pq}. \tag{14}$$

The even and odd sectors are generalizations of the Ramond and Neveu-Schwarz sectors for the case $N=2$.

The χ_{pq} can be found from (13) and (14) and with the help of the string functions[14] $c_m^l(z)$:

$$\chi_{pq}(z)=\sum_{l=0,1}^{N}\sum_{m=0}^{N}c_m^l(z)\left[\sum_{n\in Z}\delta_{mr}z^{\alpha_{pq}(n)}-\sum_{n\in Z}\delta_{ms}z^{\beta_{pq}(n)}\right]. \tag{15}$$

Here l runs over even (odd) integers for $p-q$ even (odd); $(s,r)=|(p\pm q)+2(M+2)n \pmod N|$; and

$$\alpha_{pq}(n)=\beta_{p,-q}(n)=\frac{[2(N+M+2)(M+2)n+(N+M+2)p-(M+2)q]^2-N^2}{4N(N+M+2)(M+2)}. \tag{16}$$

The χ_{pq} play the role of characters for the new symmetry algebras V. They are in one-to-one correspondence with the primary fields Ψ_{pq} introduced above. The proof that the χ_{pq} are indeed the characters of a new algebra V requires that we identify all the currents of V and check that the corresponding Verma modules agree with the characters level by level.

An important open problem is to clarify the connection between the Feigin-Fuchs and GKO realizations of these models. For example, the question of unitarity is best approached from the GKO point of view. For $N=1$ and 2, the models discussed here are unitary, as was shown in Ref. 5. We believe that our models are unitary for higher N as well. A full proof of unitarity would require the construction of the complete set of currents in terms of su(2) representations. Work along these lines is in progress.

The above construction of extended models with $c>1$ can be applied to other affine Lie algebras. For example, the GKO construction with $g=\mathrm{su}(n)\oplus\mathrm{su}(n)$ and $h=\mathrm{su}(n)$ gives rise to a series of models with central charge

$$c=\frac{(n^2-1)MN(M+N+2n)}{(M+N+n)(M+n)(N+n)}. \tag{17}$$

These models can be represented in terms of generalized parafermions[15] and $n-1$ scalar fields. The Feigin-Fuchs screening operators can be constructed from the parafermionic fields and appropriate vertex operators. The vertex operators have the form $V_{\boldsymbol\alpha}=e^{i\boldsymbol\alpha\cdot\boldsymbol\phi}$, where the $\boldsymbol\alpha$ are proportional to the roots of the su(n) algebra. The primary fields of this algebra are labeled by two $(n-1)$-dimensional vectors $\mathbf{p}$ and $\mathbf{q}$. The situation for general g and h has been recently discussed by Douglas.[16]

We would like to express our thanks to the SLAC theory group and the Aspen Center for Physics for hospitality while this work was performed. We would also like to thank Alexander Zamolodchikov and Michael Douglas for helpful conversations. After this work was completed, we received a preprint by Kastor, Martinec, and Qiu which discusses some of points presented here.[17] This work was supported by the Alfred P. Sloan Foundation, the National Science Foundation Grants No. PHY82-15249 and No. PHY86-57291, the U.S. Department of Energy Grant No. DE-FG03-84ER-40168, the U.S.-Israel Binational Science Foundation, and the Israel Academy of Science.

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