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Measurement Breaking the Standard Quantum Limit for Free-Mass Position

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An explicit interaction-Hamiltonian realization of a measurement of the free-mass position with the following properties is given: (1) The probability distribution of the readouts is exactly the same as the free-mass position distribution just before the measurement. (2) The measurement leaves the free mass in a contractive state just after the measurement. It is shown that this measurement breaks the standard quantum limit for the free-mass position in the sense sharpened by the recent controversy.

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For monitoring the position of a free mass such as the gravitational-wave interferometer,[1] it is usually supposed[2,3] that the predictability of the results is limited by the so-called standard quantum limit (SQL). In the recent controversy,[4-8] started with Yuen's proposal[4] of a measurement which beats the SQL, the meaning of the SQL has been much clarified and yet no one has given a general proof nor a counterexample for the SQL. Recently, Ni[9] succeeded in constructing a repeated-measurement scheme to monitor the free-mass position to an arbitrary accuracy. However, it is open whether this scheme beats the SQL in the sense sharpened by the recent controversy. In particular, the following problem remains open: Can we realize a high-precision measurement which leaves the free mass in a contractive state?

In the present paper, I shall give a model of measurement of a free-mass position which breaks the SQL in its most serious formulation. An explicit form of the system-meter interaction Hamiltonian will be given and it will be shown that if the meter is prepared in an appropriate contractive state[4] then the measurement leaves the free mass in a contractive state and the uncertainty of the prediction for the next identical measurement decreases in a given duration to a desired extent. Thus Yuen's original proposal[4] is fully realized. This result will open a new way to an arbitrarily accurate non-quantum-nondemolition monitoring for gravitational wave detection and other related fields such as optical communications.

The precise formulation of the SQL is given by Caves[8] as follows: Let a free mass m undergo unitary evolution during the time τ between two measurements of its position x, made with identical measuring apparatus; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$ in average over all the first readout values. Caves[8] showed that the SQL holds for a specific model of a position measurement due to von Neumann[10] and he also gave the following heuristic argument for the validity of the SQL. His point is the notion of the imperfect resolution σ of one's measuring apparatus. His argument runs as follows: *The first assumption* is that the variance of the measurement of x is the sum of σ^2 and the variance of x at the time of the measurement; this is the case when the measuring apparatus is coupled linearly to x. *The second assumption* is that just after the first measurement, the free mass has position uncertainty $\Delta x(0) \le \sigma$. Under these conditions, he derived the SQL from the uncertainty relation $\Delta x(0)\Delta x(\tau) \ge \hbar\tau/2m$.

However, his definition of the resolution of a measurement is ambiguous. In fact, he used three different definitions in his paper: (1) the uncertainty in the result, (2) the position uncertainty after the measurement, and (3) the uncertainty of the meter before the measurement. These three notions are essentially different, although they are the same for von Neumann's model. I

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shall make the following distinction: If the free mass is in a position eigenstate at the time of measurement of x then the *precision* ϵ of the measurement is defined to be the uncertainty in the result and the *resolution* σ is defined to be the deviation of the position of the free mass just after the measurement from the readout just obtained. In what follows, I shall give precise definitions for the case of superposition. For the free-mass state ψ at the time of measurement, let $P(a\,|\,\psi)$ be the probability density of obtaining the result a and let ψ_a be the state of the free mass just after the measurement. The physical design and the indicated preparation of the apparatus determine $P(a\,|\,\psi)$ and the transition (or state reduction) $\psi \rightarrow \psi_a$ for all possible ψ. These two elements will be called the *statistics* of a given measurement. This implies that if two measurements are identical then the corresponding two statistics are identical. A difficult step in defining the precision is to extract the noise factor from the readout distribution. Even if the measuring apparatus measures the position observable approximately, the readout distribution $P(a\,|\,\psi)$ should be related to the position distribution $|\psi(x)|^2$; this relation can usually be expressed in the following form:

$$P(a\,|\,\psi)=\int G(a,x)\,|\psi(x)|^2 dx, \tag{1}$$

where $G(a,x)$ is independent of a particular wave function $\psi(x)$ and expresses the noise in the readout. Obviously, $P(a\,|\,\psi)=|\psi(a)|^2$ for all ψ (i.e., the noiseless case) if and only if $G(a,x)=\delta(x-a)$. From Eq. (9) of Ref. 8, in the case of von Neumann's model, $G(a,x)=|\Psi(a-x)|^2$, where Ψ is the prepared state of the meter.[11] Roughly speaking, $G(a,x)$ is the (normalized) conditional probability density of the readout a, given that the free mass is in the position x at the time of measurement; hence the precision $\epsilon(x)$ of this case should be

$$\epsilon(x)^2=\int (a-x)^2 G(a,x)da.$$

Thus if the free mass is in a state ψ at the time of measurement, the *precision* $\epsilon(\psi)$ of the measurement is given by

$$\epsilon(\psi)^2=\int \epsilon(x)^2\,|\psi(x)|^2 dx. \tag{2}$$

By similar reasoning, for the free-mass state ψ at the time of measurement, the *resolution* $\sigma(\psi)$ of the measurement is given by

$$\sigma(\psi)^2=\int \sigma(a)^2 P(a\,|\,\psi)da,\qquad \sigma(a)^2=\int (a-x)^2\,|\psi_a(x)|^2 dx. \tag{3}$$

Let $\Delta(\psi)$ be the uncertainty of the readout for the free-mass state ψ just before the measurement and $\Delta x(\phi)$ the uncertainty of the free-mass position at any state ϕ. Suppose that the noise and state reduction of measurement are unbiased, in the sense that the mean value of the readout is identical with the mean position of the free mass just before the measurement and that the mean position of the free mass just after the measurement is identical with the readout value, i.e.,

$$\int xP(x\,|\,\psi)dx=\langle\psi\,|\,\hat{x}\,|\,\psi\rangle, \tag{4}$$

$$\langle\psi_a\,|\,\hat{x}\,|\,\psi_a\rangle=a, \tag{5}$$

for all possible ψ. Then, in general, we can prove the relations[12]

$$\Delta(\psi)^2=\epsilon(\psi)^2+\Delta x(\psi)^2, \tag{6}$$

$$\sigma(\psi)^2=\int \Delta x(\psi_a)^2 P(a\,|\,\psi)da. \tag{7}$$

Let $\epsilon_{\max}$ be the maximum of $\epsilon(\psi)$ ranging over all ψ. Then we have $\epsilon_{\max}=0$ if and only if $P(x\,|\,\psi)=|\psi(x)|^2$ for all ψ and that $\sigma(\psi)=0$ if and only if $\Delta x(\psi_a)=0$ almost everywhere [with respect to $P(a\,|\,\psi)da$]. These relations clearly show the precise meaning of the assumptions used for the derivation of the SQL by Caves. The first assumption always holds for the *precision* by Eq. (6) and the second assumption always holds for the *resolution* in average by Eq. (7). Thus his proof has shown that if the resolution is (less than or) equal to the precision then the SQL holds. In fact, under the sole condition $\sigma(\psi)^2\le[\epsilon(\tau)^2]$, where the brackets mean the average over all readouts at time 0, we can derive from the uncertainty relation the following estimate for the uncertainty $\Delta(\tau)$ of the second measurement at time τ:

$$\begin{aligned}[\Delta(\tau)^2]&=[\epsilon(\tau)^2]+[\Delta x(\tau)^2]\ge\sigma(\psi)^2+[\Delta x(\tau)^2]\\&=[\Delta x(0)^2]+[\Delta x(\tau)^2]\ge[2\Delta x(0)\Delta x(\tau)]\\&\ge\hbar\tau/m.\end{aligned}$$

Now let ΔQ be the uncertainty of the pointer position just before the measurement. Then in von Neumann's model we have

$$\epsilon=\epsilon(\psi)=\Delta Q\ \text{ for all }\psi,\qquad \sigma=\sigma(\psi)=\Delta Q\ \text{ for all }\psi. \tag{8}$$

Thus we have $\sigma(\psi)=\Delta Q=\epsilon(\tau)$ and hence the SQL holds. However, it is not at all clear that every measurement should or might satisfy the relation $\sigma(\psi)^1\le[\epsilon(\tau)^2]$.

I shall now turn to Yuen's proposal.[4] His observation is that if the measurement leaves the free mass in a contractive state ψ_a for every readout a then we can get

$$\Delta x(\tau)(\psi_a)\ll(\hbar\tau/2m)^{1/2}\ll\Delta x(0)(\psi_a). \tag{9}$$

Thus the SQL breaks if such a measurement has a good precision $\epsilon(U_\tau\psi_a)\ll(\hbar\tau/2m)^{1/2}$, where U_τ stands for the time evolution: In fact, from the combination of Eqs. (6) and (9), we get

$$\begin{aligned}\Delta(\tau)^2&=\Delta(U_\tau\psi_a)^2\\&=\epsilon(U_\tau\psi_a)^2+\Delta x(\tau)(\psi_a)^2\ll\hbar\tau/m.\end{aligned} \tag{10}$$

The model presented below has the following surprising properties: (1) If the prepared state of the apparatus is an arbitrarily chosen contractive state $|\mu\nu\tilde{a}\omega\rangle$ ($\tilde{a}=0$), then it leaves the free mass in the contractive state $|\mu\nu\tilde{a}'\omega\rangle$ ($\tilde{a}'=a$) just after the measurement for the readout value a, where[13]

$$\langle x\,|\,\mu\nu\tilde{a}\omega\rangle=\left(\frac{m\omega}{\pi\hbar\,|\mu-\nu|^2}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}\frac{1+2\xi i}{|\mu-\nu|^2}(x-x_0)^2+\frac{i}{\hbar}p_0(x-x_0)\right) \tag{11}$$

and

$$|\mu|^2-|\nu|^2=1;\quad \xi=\mathrm{Im}(\mu^*\nu)>0;\quad \tilde{a}=x_0+ip_0,\quad x_0,p_0 \text{ real.} \tag{12}$$

(2) The precision $\epsilon(\psi)$ of this measurement is such that $\epsilon(\psi)=0$, for any free-mass state ψ just before the measurement.

Thus the SQL can be broken to any extent. Further, this measurement corresponds to the measurement $|\mu\nu a\omega\rangle\langle a|$ in the terminology of Gordon and Louisell.[14]

The model description is as follows, parallel with the exposition of von Neumann's measurement by Caves; for the detailed account omitted here, see his paper.[8] The free mass is coupled to a meter which is a one-dimensional system with coordinate Q and momentum P. The coupling is turned on from $t=-\tilde{\tau}$ to $t=0$ ($0<\tilde{\tau}\ll\tau$) and it is assumed to be so strong that the free Hamiltonians of the mass and the meter can be neglected. I choose the following interaction Hamiltonian:

$$H=(K\pi/3\sqrt{3})\{2(\hat{x}\hat{P}-\hat{Q}\hat{p})+(\hat{x}\hat{p}-\hat{Q}\hat{P})\}, \tag{13}$$

where K is the coupling constant chosen as $K\tilde{\tau}=1$. Then if $\Psi_0(x,Q)=f(x,Q)$, the solution of the Schrödinger equation is

$$\Psi_t(x,Q)=f\{\tfrac{2}{3}\sqrt{3}[x\sin(\tfrac{1}{3}(1-Kt)\pi)+Q\sin(\tfrac{1}{3}Kt\pi)],\ \tfrac{2}{3}\sqrt{3}[-x\sin(\tfrac{1}{3}Kt\pi)+Q\sin(\tfrac{1}{3}(1+Kt)\pi)]\}. \tag{14}$$

At $t=-\tilde{\tau}$, just before the coupling is turned on, the unknown free-mass wave function is $\psi(x)$, and the meter is prepared in a contractive state $\Phi(Q)=\langle Q\,|\,\mu\nu0\omega\rangle$, so that the total wave function is $\Psi_0(x,Q)=\psi(x)\Phi(Q)$; expectation values for this state are $\langle\hat{Q}\rangle_0=\langle\hat{P}\rangle_0=0$. At $t=0$, the end of the interaction, the total wave function becomes

$$\Psi(x,Q)=\psi(Q)\Phi(Q-x). \tag{15}$$

[Compare with Eq. (6) of Ref. 8; the statistics are very different.] At this time one reads out a value of $\bar{Q}$ for Q by another instrument precisely, from which one infers a value for x. Then the probability density $P(\bar{Q}\,|\,\psi)$ to obtain the value $\bar{Q}$ as the result of this measurement is given by

$$P(\bar{Q}\,|\,\psi)=\int dx\,|\Psi(x,\bar{Q})|^2=|\psi(\bar{Q})|^2. \tag{16}$$

Thus the expected result is $\langle\hat{Q}\rangle=\langle\psi\,|\,\hat{x}\,|\,\psi\rangle$, and hence Eq. (4) holds. The variance of the readout coincides with the variance of x at $t=-\tilde{\tau}$:

$$\Delta_1^2=\Delta(\psi)^2=\Delta Q(\Psi)^2=\Delta x(\psi)^2 =\langle\psi\,|\,\hat{x}^2\,|\,\psi\rangle-\langle\psi\,|\,\hat{x}\,|\,\psi\rangle^2. \tag{17}$$

The free-mass wave function $\psi_{\bar{Q}}(x)=\psi(x\,|\,\bar{Q})$ just after the first measurement ($t=0$) is obtained (up to normalization) by

$$\psi(x\,|\,\bar{Q})=[1/P(\bar{Q}\,|\,\psi)^{1/2}]\Psi(x,\bar{Q})=[\psi(\bar{Q})/|\psi(\bar{Q})|]\Phi(\bar{Q}-x)=C\langle x\,|\,\mu\nu\bar{Q}\omega\rangle, \tag{18}$$

where C ($|C|=1$) is a constant phase factor. From Eq. (8) of Ref. 4, $\langle\psi_{\bar{Q}}\,|\,\hat{x}\,|\,\psi_{\bar{Q}}=\bar{Q}$ and hence Eq. (5) holds.

During the time τ until the second measurement, the free mass evolves unitarily. Observer's prediction for the mass position at time $t=\tau$ can be made as

$$\langle\psi_{\bar{Q}}\,|\,\hat{x}(\tau)\,|\,\psi_{\bar{Q}}\rangle=\langle\mu\nu\bar{Q}\omega\,|\,\hat{x}(\tau)\,|\,\mu\nu\bar{Q}\omega\rangle=\bar{Q}, \tag{19}$$

where $\hat{x}(\tau)=\hat{x}+\hat{p}\tau/m$. (One does know the wave function $\psi_{\bar{Q}}$.) The unpredictability of the second measurement is characterized by the variance of the readout [obtained by the same statistics as Eq. (17)],

$$\Delta_{2,\bar{Q}}^2=\Delta x(\tau)(\psi_{\bar{Q}})^2=\Delta x(\tau)(|\mu\nu\bar{Q}\omega\rangle)^2 =\frac{2\hbar}{m}\left(\frac{|\mu-\nu|^2}{4\omega}-\xi\tau+\frac{|\mu+\nu|^2\omega\tau^2}{4}\right). \tag{20}$$

The last equality follows from Eq. (13) of Ref. 4. Thus the desired relation has been realized; see Ref. 4 for the detailed minimization of $\Delta x(\tau)$.

It should be noticed that the general realization problem of quantum measurements was resolved in my previous paper,[15] where I proved that *every completely positive operation-valued measure has a unitary realization.* Thus, every Gordon-Louisell–type measurement has a unitary realization. In this paper I have given an explicit Hamiltonian realization of one of them.

I can say following Schrödinger[16] with a little alteration: The systematically arranged interaction of two systems is called a measurement on the first system, if a directly sensible variable feature of the second is always *predictable* within certain error limits when the process is repeated immediately *or after some arranged duration.* In the latter case, the measurement will be called a *focused measurement.*

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[11]See also, C. M. Caves, Phys. Rev. D **33**, 1643 (1986), where $\Psi(a-x)$ is called the resolution amplitude.

[12]The detailed derivation of these relations will be published elsewhere.

[13]See Ref. 4. for details about the moments of this state. The relation between our parameter $\tilde{a}$ and the parameter $a=a_1+ia_2$ in Ref. 4 is such that $x_0=(2\hbar/m\omega)^{1/2}a_1$ and $p_0=(2\hbar m\omega)^{1/2}a_2$. See also B. L. Schumaker, Phys. Rep. **135**, 317 (1986).

[14]J. P. Gordon and W. H. Louisell, in *Physics of Quantum Electronics,* edited by P. L. Kelly, B. Lax, and P. E. Tannenwald (McGraw-Hill, New York, 1966), p. 833. In their terminology, the measurement $|\mu\nu a\omega\rangle\langle a|$ is such that $P(a/\psi)=|\langle a|\psi\rangle|^2$ and that $\psi_a=|\mu\nu a\omega\rangle$.

[15]M. Ozawa, J. Math. Phys. **25**, 79 (1984), where an operation-valued measure is called an instrument.

[16]E. Schrödinger, Naturwissenschaften **23**, 807 (1935) [English translation: J. D. Trimmer, Proc. Am. Philos. Soc. **124**, 323 (1980)].