Microscopic Theory of the Proximity-Induced Josephson Effect

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The proximity-induced Josephson effect is studied microscopically by analysis of the quasiclassical transport equations in a sandwich geometry. The main Shapiro step is found at $V = hv/4e$. It is thus concluded that the proximity-induced Josephson effect is not seen in a recent experiment by Han *et al.*

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Han *et al.*[1-3] have recently made experimental studies of point contacts between a strong (s wave) superconductor (Nb,Ta) and a weaker superconductor of unknown type (UBe_{13}, $CeCu_2Si_2$, $LaBe_{13}$). Surprisingly, they found both the dc Josephson effect and the ac Josephson effect (Shaprio steps) even at temperatures considerably above the transition temperature of the weak superconductor. To interpret their observations, they introduced a theory of a "proximity-induced Josephson effect," in which the Cooper pairs leaking from the bulk superconductor not only induce a region of superconductivity in the weak superconductor, but also weakly couple the order parameters in the bulk and the induced superconducting regions, thereby permitting dc and ac Josephson effects. Subsequently, Kadin and Goldman,[4] basing their reasoning on the time-dependent Ginzburg-Landau equation, disputed this explanation and argued that the phase of the order parameter in the proximity region is intimately tied to the phase of the bulk superconductor.

In this Letter we make a microscopic theory of the proximity-induced Josephson effect. Our purpose is to check the assumptions made in the two phenomenological theories and to decide whether the proximity-induced Josephson effect can explain the experiment by Han *et al.* We make a model of the junction geometry which should at least qualitatively reproduce the properties of a real proximity-induced Josephson effect. Using the general quasiclassical theory of superconductivity, we solve the current-phase relation exactly for this model. For the dc effect, our calculation justifies the phenomenological model of Han *et al.*,[1] but is in clear disagreement with Kadin and Goldman.[4,5] In particular, we find that there *can* be a nonzero difference of phase between the bulk and proximity-induced order parameters in equilibrium, and that the current-phase relation is of the form $J_s = J_c \sin(2\phi)$, i.e., it has *twice* the phase difference ϕ in place of ϕ in the conventional Josephson effect. We then give arguments to show that in the ac Josephson effect Shapiro steps occur at voltages $V_n = nhv/4e$ instead of the regular voltages $V_n = nhv/2e$. [The analysis of Ref. 1 erroneously gives the regular Shapiro steps because it supposed that the current-phase relation $J_s = J_c \sin(2\phi)$ becomes invalid if $|\phi| > \pi/2$.] Since steps corresponding to the regular Josephson frequency were seen in the experiments by Han *et al.*, we conclude that the experiments[1,2] cannot be explained by a proximity-induced Josephson effect. A possible explanation is that the observed Shapiro steps originate from a regular Josephson effect between two superconducting regions which both have higher transition temperatures than the observation temperature.

The effect we are predicting should be experimentally observable.

There is a simple physical explanation for the double frequency of the Josephson current-phase relation. Since the weak superconductor (denoted by S, order parameter Δ) is above its transition temperature T_c, its superconductivity is wholly supported by the strong superconductor (S', order parameter Δ', transition temperature T_c') and it is localized near their contact. This implies that the maximum phase difference between the order parameters is $\pi/2$ because otherwise the strong superconductor tends to destroy the proximity superconductivity in S. More specifically, the calculation gives $|\Delta| \sim \cos\phi$, and when combined with the standard Josephson relation $J_s \sim |\Delta| \, |\Delta'| \sin\phi$, it gives $J_s \sim \sin(2\phi)$. An interesting situation arises when ϕ approaches $\pm\pi/2$ because the order parameter vanishes there. This special point is discussed after the model calculation.

Our model is the following: In addition to two pure superconductors S and S' and a weakly transmitting barrier between them, we add another barrier on the weak superconducting side; see Fig. 1. The part of the weak superconductor not contained between the barriers is denoted by N and, in order to avoid introducing voltage gradients, we take it to have an infinite conductivity. We assume two small ratios: (1) The distance between the barriers D is small compared to the coherence length $\xi_{\text{prox}} = \hbar v_F / 2\pi k_B T$, and (2) the transmission probability τ through both barriers is small compared to D/ξ_{prox}. The former condition allows one to neglect the variation of the order parameter in the proximity region S. The latter implies that a particle entering S will stay there for a long distance compared to ξ_{prox}. For the following calculation several of the assumptions can be relaxed and, we believe, none of them is crucial for our qualitative conclusions.

The problem can be solved at all temperatures with the standard quasiclassical theory.[6] The transportlike

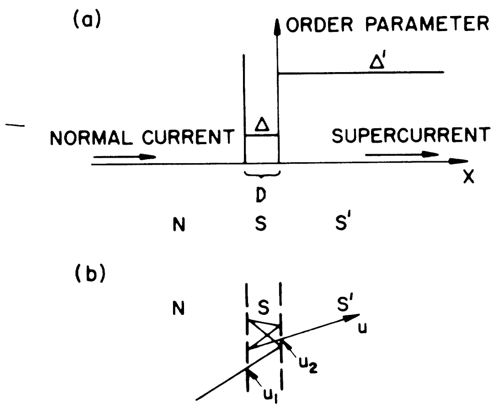

FIG. 1. (a) The order parameters and currents and (b) one representative trajectory in the model of a proximity-induced Josephson effect. The calculation presented here is an exact expansion to leading order in the small parameters D/ξ_{prox} and $\xi_{\text{prox}}/(u_2-u_1)$.

equation reads

$$[i\epsilon_n \hat{\tau}_3 - \hat{\Delta}, \hat{g}^{\text{M}}(\hat{\mathbf{k}},\mathbf{R};\epsilon_n)] + iv_{\text{F}}\hat{\mathbf{k}}\cdot\nabla\hat{g}^{\text{M}}(\hat{\mathbf{k}},\mathbf{R};\epsilon_n) = 0 \quad (1)$$

for the 2×2 matrix Matsubara propagator $\hat{g}^{\text{M}}(\hat{\mathbf{k}},\mathbf{R};\epsilon_n)$, where $\epsilon_n = \pi(2n+1)T$. For a time-independent problem, the retarded, advanced, and Keldysh propagators satisfy the same equation with $i\epsilon_n$ replaced by ϵ. The normalization condition $\hat{g}\hat{g} = -\pi^2$ applies to $\hat{g}^{\text{M}}$, $\hat{g}^R$, and $\hat{g}^A$, but for $\hat{g}^{\text{K}}$ one has $\hat{g}^R\hat{g}^{\text{K}} + \hat{g}^{\text{K}}\hat{g}^A = 0$. Equation (1) forms a closed set together with the weak-coupling gap equation

$$\Delta(\mathbf{R}) \equiv \hat{\Delta}_{12}(\mathbf{R}) = V_{\text{BCS}}(\mathbf{R})\frac{d\Omega_k}{4\pi}\int\frac{d\epsilon}{4\pi i}\hat{g}^{\text{K}}_{12}(\hat{\mathbf{k}},\mathbf{R};\epsilon). \quad (2)$$

Here the subscript ij denotes the component of the Nambu matrix. The coupling constant V_{BCS} is chosen smaller in S than in S' to produce the required T_c's. We are interested in the temperature region between the transition temperatures ($T_c < T < T_c'$). Equation (1) has the intuitive feature of being an ordinary differential equation along straight lines ($\mathbf{R} = \mathbf{R}_0 + u\hat{\mathbf{k}}$, where u is the distance along the trajectory). Equations (1) and (2) have to be supplemented by a boundary condition specifying what happens when a trajectory hits a surface. We use the simple boundary condition that the "particle" [solution of Eq. (1)] is transmitted with probability τ and reflected with probability $1-\tau$ on the barriers.[7] This is equivalent to the assumption that the barriers are completely insulating except for small "holes." One representative trajectory of a particle is sketched in Fig. 1(b). The electric current can be calculated from this formula

$$j(\mathbf{R}) = 2ev_{\text{F}}N(0)\int\frac{d\Omega_k}{4\pi}\int\frac{d\epsilon}{4\pi i}\hat{\mathbf{k}}\hat{g}^{\text{K}}_{11}(\hat{\mathbf{k}},\mathbf{R};\epsilon). \quad (3)$$

The plan is to solve the propagator from Eq. (1) with a general form of $\hat{\Delta}$ and then apply (2) to determine $\hat{\Delta}$ self-consistently. Because of assumptions 1 and 2, we can take the order parameters to be constant in both S and S'. Without loss of generality we take the order parameter in S' to be real ($\hat{\Delta}' = i\Delta'\hat{\tau}_2$) and in S we use the general form [$\hat{\Delta} = i(\Delta_1\hat{\tau}_1 + \Delta_2\hat{\tau}_2)$, $\Delta_1 = \text{Im}\Delta$, $\Delta_2 = \text{Re}\Delta$]. Because of assumption 2, the transmissions of both barriers can be calculated separately. The general propagator can be represented as $\hat{g} = \sum g_i\hat{\tau}_i$, where the $\hat{\tau}_i$'s are the Pauli matrices.

Let us first study the S-S' junction. For a spatially constant gap, Eq. (1) has three independent solutions: one constant, one exponentially growing, and one exponentially decreasing.[8] Matching these to the boundary conditions gives the Matsubara propagator in S:

$$\hat{g}^{\text{M}} = \frac{i}{\alpha}\begin{pmatrix}\Delta_1\\ \Delta_2\\ -\epsilon_n\end{pmatrix} + B\tau\begin{pmatrix}-\Delta_2\alpha s_k + i\Delta_1\epsilon_n\\ \Delta_1\alpha s_k + i\Delta_2\epsilon_n\\ i|\Delta|^2\end{pmatrix}\exp[2\alpha(u-u_2)/v_{\text{F}}], \quad B = \frac{\epsilon_n\Delta'\Delta_2 - \epsilon_n|\Delta|^2 + is_k\alpha\Delta'\Delta_1}{\alpha|\Delta|^2(\epsilon_n^2 + \alpha\alpha' + \Delta'\Delta_2)}. \quad (4)$$

Here u is the distance along the trajectory and the junction is at $u = u_2$ [$u < u_2$ on S, Fig. 1(b)], $\alpha = (\epsilon_n^2 + |\Delta|^2)^{1/2}$, $\alpha' = (\epsilon_n^2 + \Delta'^2)^{1/2}$, and s_k is plus (minus) unity when the momentum is parallel (antiparallel) to the trajectory. The retarded and advanced propagators can be obtained by analytic continuation. Because this junction is in equilibrium, the Keldysh propagator is given by[6]

$$\hat{g}^{\text{K}} = -2[f(\epsilon) - \tfrac{1}{2}](\hat{g}^R - \hat{g}^A), \quad (5)$$

where $f(\epsilon)$ is the Fermi distribution.

In the N-S junction (junction at $u = u_1$), the Matsubara propagator on the side S is given by

$$\hat{g}^{\text{M}} = \frac{i}{\alpha}\begin{pmatrix}\Delta_1\\ \Delta_2\\ -\epsilon_n\end{pmatrix} + C\tau\begin{pmatrix}\Delta_2\alpha s_k + i\Delta_1\epsilon_n\\ -\Delta_1\alpha s_k + i\Delta_2\epsilon_n\\ i|\Delta|^2\end{pmatrix}\exp[-2\alpha(u-u_1)/v_{\text{F}}], \quad C = -s_\epsilon/\alpha(\alpha + |\epsilon_n|). \quad (6)$$

The retarded and advanced propagators can be obtained by analytic continuation. In the Keldysh function, we allow the distribution function $f(\hat{k},\epsilon)$ to have a small deviation from the equilibrium $\delta f(\hat{k},\epsilon)$ in order to represent the incoming normal current. Physically, all the quasiparticles injected at energies below the gap Δ are converted into supercurrent and the same number of holes are Andreev-reflected back.[9] No energy relaxation is needed here because energy is conserved in such a process. Contrary to Refs. 4 and 5, no charge imbalance will develop. Quasiparticles having an energy greater than Δ are neglected because their contribution to the phase-dependent current [Eq. (10) below] is a small correction ($\sim \tau^3$).

We now apply the self-consistency equation (2) in S. Neglecting terms higher than first in quantities Δ_1, Δ_2, and τ, we obtain the simple equations

$$\Delta_1 \ln t + \tilde{j}\Delta_2/|\Delta|^2 = 0, \quad \Delta_2 \ln t - \tilde{j}\Delta_1/|\Delta|^2 - \beta = 0. \tag{7}$$

Here $t = T/T_c > 0$,

$$\tilde{j} = \frac{v_F \tau}{2D}\int d\epsilon \int \frac{d\Omega_k}{4\pi}\hat{\mathbf{k}}\cdot\hat{\mathbf{x}}\,\delta f(\hat{\mathbf{k}},\epsilon),$$

$$\beta = \frac{v_F \tau}{4D}\pi T \sum_{\epsilon_n} \frac{\Delta'}{|\epsilon_n|[(\epsilon_n^2+\Delta'^2)^{1/2}+|\epsilon_n|]}.$$

Physically, $\tilde{j}$ is proportional to the current [$j = 4eD \times N(0)\tilde{j}$] and β describes the supporting effect of the bulk superconductor S'. If we write the order parameter in S as

$$\Delta = \Delta_2 + i\Delta_1 = A\exp(-i\phi), \tag{8}$$

Eq. (7) has the solution

$$\sin(2\phi) = 2\tilde{j}\ln t/\beta^2, \quad A = (\beta/\ln t)\cos\phi. \tag{9}$$

This solution is depicted in Fig. 2. The current as a function of the phase is given by

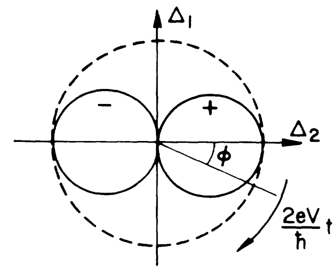

FIG. 2. Diagram illustrating the proximity Josephson effect. As the phase increases, the order parameter in S is given by the intersection of the straight line with the two circles. On the right (left) circle the order parameter is plus (minus) $\Delta_2 + i\Delta_1$, in accordance with Eq. (7). For comparison, the solution for a regular Josephson junction is indicated by the dashed circle.

$$j(\phi) = \frac{eN(0)v_F^2\tau^2}{8D\ln t}\left[\pi T \sum_n \frac{\Delta'}{|\epsilon_n|[(\epsilon_n^2+\Delta'^2)^{1/2}+|\epsilon_n|]}\right]^2 \sin(2\phi), \tag{10}$$

as claimed in the introduction. Note that the current conservation in the system is implicit in Eqs. (1) and (2) and no separate conservation equation was needed.

It is useful to calculate the free-energy functional of the system. From the general functional of the quasiclassical theory,[6] one can derive the following differential equation:

$$dG = 2N(0)\int d^3R\,\mathrm{Re}\left[\left(\frac{\Delta}{V_{\mathrm{BCS}}} - \int\frac{d\Omega_k}{4\pi}\int\frac{d\epsilon}{4\pi i}\hat{g}_{12}^K\right)d\Delta^*\right]. \tag{11}$$

The part in the parentheses is the self-consistency equation which for the present system is given by the left-hand sides of (7). Integrating (11), one obtains the free energy functional

$$G(A,\phi) = 2DN(0)[\tfrac{1}{2}A^2\ln t - \beta A\cos\phi - \gamma\phi]. \tag{12}$$

This justifies the phenomenological theory of Ref. 1 because the first two terms are just the same as there. The last term describes the energy of the current source in a current-drive system.

To deduce the ac Josephson effect we argue as follows. The phase and the amplitude of the order parameter (8) are physically quite different: In a voltage-driven junction the phase follows the external voltage difference according to the standard relation $d\phi/dt = 2eV/\hbar$.[10] In a current-driven system, the phase is determined by an equation where the capacitance of the junction plays the role of a mass and the resistance describes dissipation. In contrast, the amplitude has no inertia and for any ϕ it instantly adjusts itself to the ϕ-dependent equilibrium value (if the rate of change of ϕ is slow compared to $\Delta/\hbar$). This equilibrium value crosses zero at $\phi = \pm\pi/2$ according to Eq. (9). This solution seems physical to us because once ϕ is outside the region $|\phi| > 2\pi/2$, the bulk superconductivity supports a proximity-induced order-parameter amplitude with the opposite sign. Therefore we conclude that Eqs. (9) and (10) are valid at all values of ϕ. When this is combined with $d\phi/dt = 2eV/\hbar$, one gets the main Shapiro step at $V = h\nu/4e$. This is in contrast to Ref. 1 where it was incorrectly assumed that the amplitude has to remain positive, leading to the regular Shapiro step ($V = h\nu/2e$).

We should add the following qualification to the result of the ac Josephson effect. We have not been able to rule out the possibility that the system, having reached the point $|\Delta| = 0$, recovers through some fundamentally nonequilibrium way to finite Δ. If this were the case, it would give the Shapiro step at even lower voltage and therefore it does not change our conclusion that the proximity-induced Josephson effect is not seen by Han *et al.*

The proximity-induced Josephson effect exists in principle in all Josephson junctions in which the transition temperatures of the two superconductors are not the same, at least when the transmission probability τ is small. The equation for the current (10) should be qualitatively correct when D (the thickness of our model slab) is replaced by the temperature-dependent coherence length. The proximity critical current is smaller than the regular Josephson current because the former is proportional to τ^2 whereas the latter is linear in τ.

We conclude with some brief comments on the analysis of Refs. 2 to 5. First, the calculations of Refs. 2 and 3 on the proximity model are not relevant to the observations because, as we have shown, these cannot be explained by the proximity effect. The main conclusion of Ref. 2 that the bulk superconductivity in UBe_{13} is hostile to the superconductivity found above T_c may remain true, but the quantitative calculation supporting this conclusion (solid line of Fig. 2 in Ref. 2) breaks down. Second, the theory of Ref. 5 fails because it neglects the continuity equation for the current, as was already pointed out in the Response of Wolf, Millis, and Han.[11]

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