

Comment on “Bistability Driven By Weakly Colored Gaussian Noise: The Fokker-Planck Boundary Layer and Mean First-Passage Times”

In a recent Letter, Doering, Hagen, and Levermore¹ made an attempt to evaluate the mean first-passage time (MFPT) for the archetypal bistability: $\dot{x} = x - x^3 + u$, $\dot{u} = -\epsilon^{-2}u + \sigma\epsilon^{-2}\xi(t)$, with $\xi(t)$ being white Gaussian noise, i.e., $\langle \xi(t)\xi(s) \rangle = 2\delta(t-s)$. This flow implies colored noise $\langle u(t)u(0) \rangle = (\sigma/\epsilon)^2 \exp(-|t|/\epsilon^2)$. The authors of Ref. 1 make use of a scaling $z(t) = \epsilon u(t)/\sigma$ and proceed to evaluate at *weak noise* ($\sigma^2 \ll 1$) “the MFPT T of the variable $x(t)$ from the well at $x=1$ to the unstable point $x=0$.” They do not use a MFPT approach based on the adjoint operator, but use instead a method based on a steady-state density per unit flux, $G(x,z)$. For $G(x,z)$ they correctly use the boundary condition $G(0,z) = 0, z > 0$ (dash-dotted line in Fig. 1). We wish to point out, however, that this boundary condition is not suitable to evaluate the activation rate to leave the *domain of attraction* or, equivalently, the smallest nonvanishing eigenvalue of the two-dimensional Fokker-Planck dynamics. Figure 1 depicts the flow for the deterministic equation [$\xi(t) = 0$] with $\epsilon^2 = 0.2$. The flow exhibits an inversion symmetry. Note that the separatrix (dotted line) has a tilt and does not coincide with the line $x=0$. Particles starting at $(1,z)$ will reach $(x=0, u > 0)$ only after having crossed the separatrix [MFPT ($x=1 \rightarrow$ separatrix) T_0] at *negative* u values. After crossing the separatrix they will settle near $x=-1$ [MFPT ($x=1 \rightarrow x=-1$) $2T_0$] before crossing the line $x=0, u > 0$. Doering, Hagen, and Levermore¹ now integrate $G(x,z)$ over the positive half plane $x > 0$ only. For $x > 0$, one has regions of integration with a MFPT [$x=1 \rightarrow x \cong 0, u = O(\sigma)$] $T_1 \gtrsim T_0$, and a MFPT ($x=1 \rightarrow$ hatched region in Fig. 1) $T_1 = 2T_0$, respectively. Near $x=0, u \lesssim 0$ (critical region), the boundary layer for the MFPT T_0 , of width $O(\sigma)$, cuts the line $x=0$, thereby yielding a *mixed contribution* to T in Eq. (8) of Ref. 1, i.e., $T_0 \leq T < 2T_0$. The final result in Eq. (23) of Ref. 1 predicts for T a growth proportional to ϵ .

We have performed precise numerical calculations (error $< 0.1\%$) for the smallest eigenvalue $\lambda(\epsilon)$ [$= T_0^{-1}(\epsilon), \sigma^2 \ll 1$] of the two-dimensional Fokker-Planck process. In the inset of Fig. 1 this eigenvalue is compared with the inverse MFPT $T(\epsilon)$ of Doering, Hagen, and Levermore to reach the line $x=0$, i.e., for the sake of illustration we plot $\lambda_D(\epsilon) \equiv T^{-1}(\epsilon)$ for small

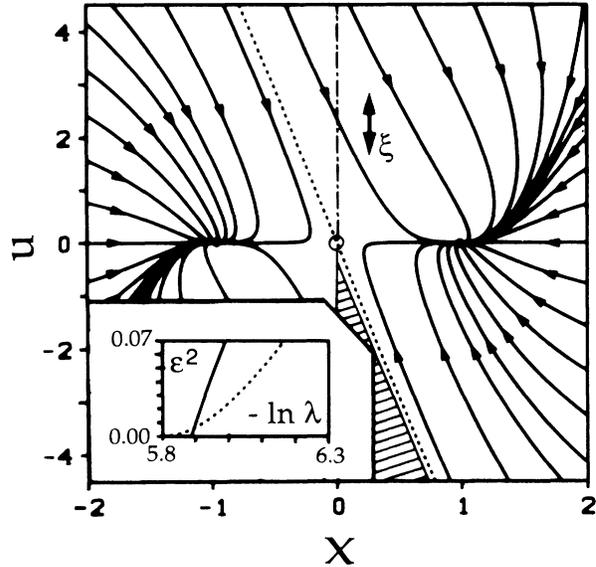


FIG. 1. Bistable flow diagram $\dot{x} = x - x^3 + u$, $\dot{u} = -\epsilon^{-2}u$ with $\epsilon^2 = 0.2$. Inset: Smallest nonvanishing eigenvalue of the two-dimensional bistable flow for $\sigma^2 = 0.05$. Solid line, exact result; dotted line, $T^{-1}(\epsilon^2)$ from Eq. (23) in Ref. 1.

values of the correlation time, ϵ^2 , of the noise.

In agreement with the above argumentation, this MFPT $T(\epsilon)$ in Eq. (23) of Ref. 1 does not compare with the inverse rate of escape or the MFPT T_0 to reach the separatrix. As exhibited in the inset of Fig. 1, the eigenvalue $\lambda(\epsilon)$ does not show a growth proportional to ϵ , i.e., $\lambda(\epsilon) \neq \lambda_D = \lambda_0(1 - \text{const} \times \epsilon)$.

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¹C. R. Doering, P. S. Hagen, and C. D. Levermore, Phys. Rev. Lett. **59**, 2129 (1987).