High-Resolution Lattice-Gas Simulation of Two-Dimensional Turbulence

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(Received 14 December 1987)

The mechanisms of two-dimensional turbulence are investigated by means of a very high-resolution lattice-gas simulation. The results from this simulation are quantitatively compared with the direct integration of the Navier-Stokes equation. The dissipation of the flow simulated by the lattice gas is estimated by a simple scaling argument on the microscopic noisy field of the automaton.

PACS numbers: 47.25.−c, 05.40.+j, 61.20.Ja

The way that the physical laws governing the evolution of macroscopic phenomena emerge out of microdynamical equations is one of the outstanding questions of modern physics and notably the core of statistical mechanics. The task of bridging the gap between the microworld and the macroworld is usually achieved by the identification of a small number of macroscopic variables and description of their evolution by an averaging over ensembles of a large number of microscopic variables. Along this line, a number of powerful analytical and numerical techniques have been developed,[1] but a number of open problems (usually characterized by the simultaneous presence of disparate scales) still remains: The statistical nature of two-dimensional turbulence is one of these.

In recent times, a new class of models, known as lattice-gas automata,[2-4] has gained considerable attention as a potential tool to address the above-mentioned problems. In particular, Frisch, Hasslacher, and Pomeau[5] (FHP) have shown that the macroscopic behavior of a system of *pseudoparticles* moving with unit speeds in a discrete hexagonal lattice is governed by the Navier-Stokes equations. Since that paper, a great deal of work has been performed and now a variety of hydrodynamical phenomena can be simulated with lattice gases.[6] However, these simulations have been usually constrained to low resolution (Reynolds number ~ 100) because of the limitations imposed by computer storage. In this Letter we present the results obtained by running the FHP-III automaton on a much higher-resolution (8192^2) seven-bit lattice containing half a billion discrete variables. To our knowledge, this is a lattice-gas regime never explored so far.

The computational domain is a square with side 2π and periodic boundary conditions so that the wave numbers are natural integers; the average Mach number is $M=0.20$ and the density per cell is $d=0.25$. The initial Boolean field of the automaton is a stochastic realization[7] at time 30 of a high-resolution (512^2) spectral simulation of decaying turbulence.[8] This is obtained by generation of the Boolean field according to a Fermi-Dirac distribution whose mean value is interpolated to the 8192^2 grid of the automaton from the original 512^2 velocity field. In this way, each value of the original velocity field generates a "population" of $b\times 16^2$ bits ($b=7$ being the number of bits per site) which represent its actual stochastic realization.

The length conversion between the units of the present simulation and the natural units of the automaton (as in Frisch *et al.*[7]) is $\alpha\equiv l/l_{\rm aut}=2\pi/8192=7.7\times 10^{-4}$. The velocity conversion as given by our initial conditions is $\eta\equiv u/u_{\rm aut}=7.9$. The time conversion is consequently determined as $\beta\equiv t/t_{\rm aut}=\alpha\eta^{-1}g=3.8\times 10^{-5}$, $g\sim 0.39$ being the characteristic factor associated with the lack of Galilean invariance of the automaton.[7] Time evolution proceeds for 4×8192 automaton steps corresponding to about 1.2 time units of the spectral code. The computation takes about 60 Mbytes of storage at a central processing unit (CPU) time cost of 6 s per automaton time step for an in-core code on the IBM model 3090 Vector Multiprocessor.[9] The fluid flow simulated by the cellular automaton is compared with the two-dimensional flow

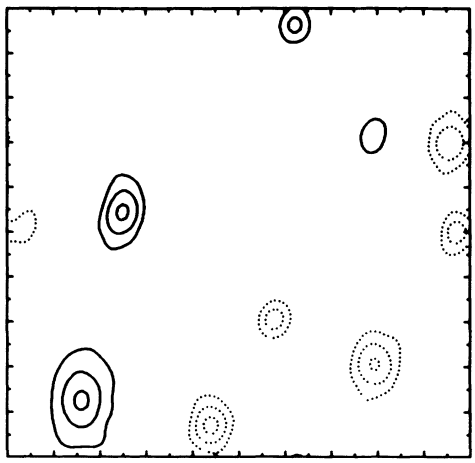

FIG. 1. The vorticity field of the automaton at the end of the simulation. The figure is obtained by our retaining only the smallest (wave number ≤ 32) components of the harmonic analysis of the Boolean field, corresponding to an average over blocks of 128^2 automaton sites. The contours are 2.5, 7.5, and 15 (solid lines) and −2.5, −7.5, and −15 (dotted lines).

obtained by the spectral code. The initial field configuration is characterized by several coherent vortices whose motion is well described in terms of long-range vortex-vortex interaction.[8] These vortices are preserved in the course of the evolution as shown in Fig. 1, which reports the vorticity contours after 4×8192 automata steps. In Figs. 2(a) and 2(b) we compare the trajectories of these vortices as obtained by the lattice-gas simulation and by the spectral code, respectively. Two major remarks should be stressed. First, the very existence of these trajectories in the lattice-gas simulation indicates that the nonlinear convective terms are strong enough and sufficiently well represented to preserve the coherent structures against dissipation. Second, the agreement between the two sets of trajectories is by far more than qualitative indicating that nonlinear long-range interactions are well reproduced. From spectral simulations of two-dimensional turbulent flows, we know that inside a coherent vortex we have $\omega=\lambda_i\psi$ (where ω is the vorticity field inside the ith vortex, ψ is the stream function, and λ_i is a constant roughly proportional to the inverse of the area of the vortex[8]). In the absence of any external forcing, as in our case, the average vorticity in a vortex decays like $e^{-\gamma_i t}$, where $\gamma_i=\nu\lambda_i$ and ν is the kinematic viscosity. The values of γ_i and λ_i are directly estimated from the simulation; for the largest vortices they provide independent estimates of the viscosity ($\nu_{\rm est}=3.9\times10^{-3}$, 3.8×10^{-3}, 4.5×10^{-3}, ...). These estimates of the viscosity compare well with the theoretical value $\nu=\alpha^2\beta^{-1}\nu_{\rm aut}=4.7\times10^{-3}$, where $\nu_{\rm aut}=0.31$ is the value given by the fluctuation-dissipation theorem.[7] The Reynolds number of the simulation $N_{\rm Re}=ul/\nu\sim500$ is estimated in terms of the maximum flow speed $u=2.5$ and the average wave number $\langle k\rangle=2\pi/l=6$ where the average is weighted with the energy spectrum $E(k)$ defined by $\int E(k)dk=\pi\langle u^2\rangle$.

From the above considerations, it turns out that the lattice-gas simulation discussed here is able to reproduce

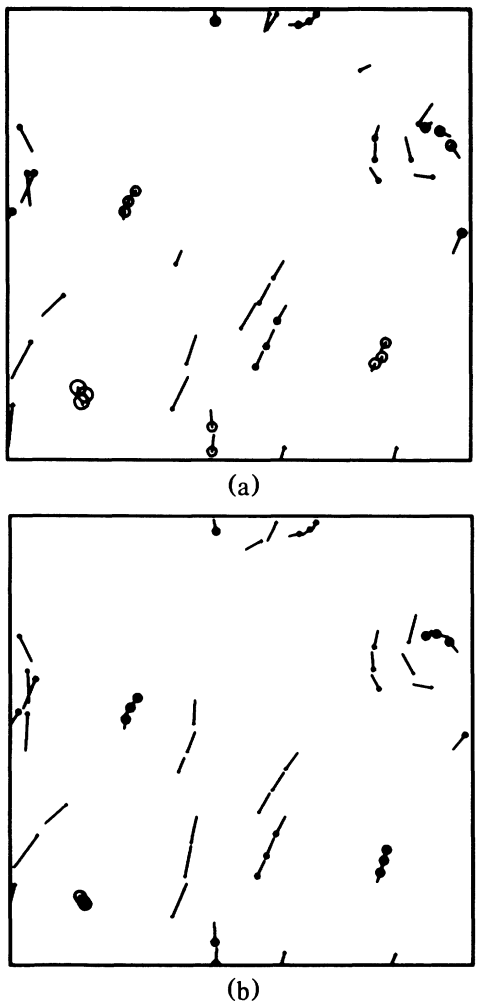

FIG. 2. Trajectories of the centers of the vortices (a) from the automaton and (b) for the spectral simulation at times $t=0$, 0.5, and 1.0 (in spectral units). The vortices are selected as those connected domains with $|\omega|\geq2.5$ and area ≥0.02. The agreement among the trajectories is remarkable despite the different spatial resolution of the two models. This is because the trajectories of large-scale vortices are almost independent of the presence of small-scale vortices (Ref. 8). Only the trajectories of the largest vortices are shown since their decay times (about 30, 12, and 10 spectral units for the largest three) are much longer than the total simulation time (1.2 spectral units). The smallest vortices have a decay time comparable to or smaller than the total simulation time and disappear from the flow in the early stage of their trajectories.

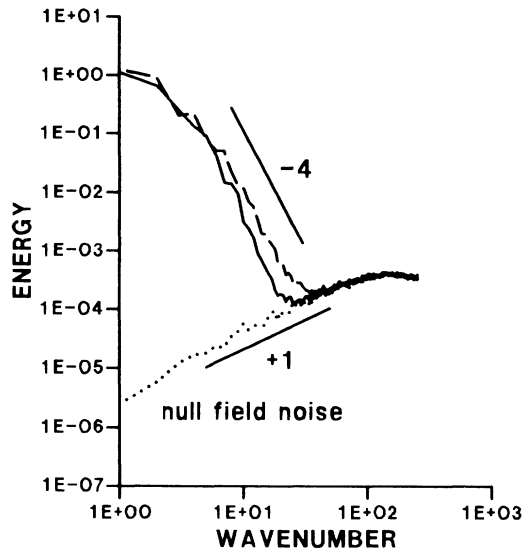

FIG. 3. Energy spectrum $E(k)$ of the automaton flow at the beginning (dashed line) and at the end (solid line) of the simulation. The dotted line is the stochastic representation of the null velocity field. The flow shown in Fig. 1 contains all the hydrodynamical scales while almost excluding the noisy component.

the full range of dynamical scales: from the long-range nonlinear Navier-Stokes dynamics to the small-scale dissipation effects embedded in the noisy "molecular" fluctuations of the automaton. The latter point is clearly shown by the energy spectrum $E(k)$ of the system (see Fig. 3). At low wave numbers we observe the k^{-4} slope of the initial configuration that changes only slightly at later times (recall that the presence of many coherent vortices produce a spectral slope steeper[8,10] than the one predicted by the Batchelor-Kraichnan theory[11]). At high k a linear increase of the spectrum is observed due to the random noise contained in the Boolean field: The white-noise scaling law is $E_n(k)=Bk$ with $B=2\pi\eta^2\sigma^2/8192^2$. In this formula the single-site variance of the automaton is given by $\sigma^2=(b-1)(\langle n^2\rangle-\langle n\rangle^2)/b^2d^2$, where $\langle n\rangle=d$ and $\langle n^2\rangle=d$, the right-hand side of these two expressions being the same since n, the occupation number of a single state of the automaton, can only take the value 0 or 1.

The two regimes are separated by a minimum whose corresponding wave number k_c marks the smallest hydrodynamical scale resulted in the system. This can be estimated by the equating of the initial hydrodynamical spectrum Ck^{-4}, with $C\sim 50$, and the noise spectrum $E_n(k)$. This gives $k_c=(C/B)^{1/5}\sim 30$ in good agreement with Fig. 3. Physically, k_c can be interpreted as the dissipative scale where the hydrodynamical signal is lost in the noise. The viscosity governing this dissipative process can be estimated on the assumption that $N_{\mathrm{Re}}(k_c)=1$, i.e., $\nu=2\pi u(k_c)/k_c$ where $u(k_c)$ is the velocity fluctuation at scale k_c. We obtain $\nu=2(\pi B/3)^{1/2}\sim 3.0\times 10^{-3}$ which compares well with the theoretical estimate based on the fluctuation-dissipation theorem[7] and also with the decay-rate of the largest coherent structures of our numerical experiment. Because of dissipative effects, k_c decreases in time as observed in Fig. 3.

Our results refer to a single experiment and therefore it is reasonable to expect that an ensemble average over many dynamical histories starting from different stochastic realizations of the initial conditions would yield better (i.e., less noisy) simulation results. However, since the viscosity of the automaton is basically controlled by the lattice size, it is clear that the achievement of substantial improvements, like, for example, an increase of the value of k_c, depends only on the capability of enhancing the spatial resolution.

In order to estimate the computational efficiency of the present lattice-gas simulation, one needs to evaluate its effective spatial resolution. By inspection of the energy spectrum one sees that the highest resolved wave number is ~ 32 which corresponds to an effective spatial resolution of $\sim 64^2$. We find that the computational cost and the storage requirements of a 64^2 spectral simulation (physical time span being the same) are about 10^4 and 10^2 times smaller than the corresponding quantities for our lattice-gas simulation. As a result, the present numerical investigation supports the conjecture that lattice gases are not computationally efficient in the simulation of turbulent flows.[12] However, it is important to remark that the resort to more efficient collision rules (such as those given by the pseudo-4D face-centered hypercube scheme[13]) and special-purpose machines can reduce the computational cost of the lattice-gas simulation by a factor 10^3 in CPU time (a factor ~ 30 for collision rules and about the same for the special-purpose hardware) and ~ 10 in terms of memory savings. These are significant improvements, especially if one considers fluid-flow simulations in complex geometries where the traditional techniques may become computationally much more demanding than for the simple case of periodic boundary conditions considered here. In any event, independent of any consideration of computational efficiency, we believe that a major point of lattice gases is that they properly describe a variety of different physical scales and provide a valuable tool to gain new physical insights into the dissipative nature of fluid flows.

We are grateful to P. Lallemand, D. d'Humières, G. F. Carnevale, and U. Frisch for valuable hints and discussions.

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