Quantum Correlations of Phase in Nondegenerate Parametric Oscillation

M. D. Reid
Physics Department, University of Waikato, Hamilton, New Zealand

and

P. D. Drummond
Physics Department, University of Auckland, Auckland, New Zealand
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The squeezing spectrum for nondegenerate parametric oscillation above threshold is calculated, including phase diffusion. A *nonclassical* correlation in phase *and* intensity occurs which is an example of the Einstein-Podolsky-Rosen paradox, even in fields of large photon number.

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There has been recent interest, both experimental and theoretical, in the various quantum features displayed by the fields of an optical nondegenerate parametric oscillator. The correlation of photon number between the signal and idler modes is greater than that predicted by standard classical theory. There is thus violation of a classical Cauchy-Schwartz inequality involving intensity correlations.[1] Graham[2] has suggested that such quantum correlations might remind one of the Einstein-Podolsky-Rosen (EPR) paradox.[3] Reynaud, Fabre, and Giacobino[4] have recently calculated the spectrum of fluctuations in the signal and idler intensity difference, and shown perfect intensity correlation to be possible at zero frequency for fields transmitted through a single-ported cavity.

We point out that recent experimental advances[5] allowing measurement of fluctuations in field quadrature-phase amplitudes at noise levels below the quantum limit now make possible a study of quantum correlations between the signal and idler phase. We calculate the transmitted spectrum of fluctuations in the difference between signal and idler quadrature amplitudes including the effect of phase diffusion in the signal and idler modes above threshold. The noise reduction possible for particular choices of quadrature-phase amplitudes indicates a perfect correlation of both intensity and phase between signal and idler. We point out that such a quadrature-phase-correlation experiment (unlike an intensity-correlation measurement alone) is a direct example of the EPR paradox, and the effect occurs both above and below threshold.

We introduce boson operators a_1, a_2, and a_3 for the signal, idler, and pump cavity modes, respectively. Using a generalized P representation, one is able to establish a correspondence between stochastic amplitudes α_i (and $\alpha_i^\dagger$) and mode operators a_i (and $a_i^\dagger$), respectively, $\alpha_i^\dagger$ and α_i being independent complex variables. Stochastic equations describing the oscillator are[1]

$$\begin{aligned}\dot{\alpha}_1 &= -\kappa_1\alpha_1 + g\alpha_3\alpha_2^\dagger + (g\alpha_3)^{1/2}\xi_1(t),\\ \dot{\alpha}_2 &= -\kappa_2\alpha_2 + g\alpha_3\alpha_1^\dagger + (g\alpha_3)^{1/2}\xi_2(t),\\ \dot{\alpha}_3 &= -\kappa_3\alpha_3 + E - g\alpha_1\alpha_2,\end{aligned} \tag{1}$$

and also the equations obtained by interchange of κ_i with κ_i^*, α_i with $\alpha_i^\dagger$, and ξ_i with $\xi_i^\dagger$. The $\xi_i(t)$ are δ-correlated noise forces with zero mean and nonzero correlation $\langle \xi_1(t)\xi_2(t')\rangle = \delta(t-t')$. E (taken real for convenience) is the input pump amplitude, and g is the nonlinear coupling due to the medium. The κ_i are the cavity damping rates and we assume $\kappa = \kappa_1 = \kappa_2$.

To study the behavior above threshold, we transform to phase and intensity variables, defined as follows:

$$\begin{aligned}&\delta I_\pm = (I_1 - I_1^0) \pm (I_2 - I_2^0),\\ &\delta I_3 = I_3 - I_3^0, \quad \delta\phi_+ = \phi_1 + \phi_2,\\ &\delta\phi_3 = \phi_3, \quad \phi_- = \phi_2 - \phi_1,\end{aligned}$$

where $I_i = \alpha_i^\dagger \alpha_i$, $\phi_i = \ln(\alpha_i^\dagger/\alpha_i)/2i$, and I_i^0 and ϕ_i^0 are the steady-state deterministic solutions with $\phi_3^0 = \phi_1^0 + \phi_2^0 = 0$. The resulting equations, after linearization in $\delta I_\pm$, δI_3, $\delta\phi_+$, and $\delta\phi_3$, are

$$\begin{aligned}\partial\delta I_+/\partial t &= 2g^2(I^0/\kappa)\delta I_3 + F_+^0(t),\\ \partial\delta I_3/\partial t &= -\kappa_3\delta I_3 - \kappa\delta I_+,\\ \partial\delta I_-/\partial t &= -2\kappa\delta I_- + F_-^0(t),\\ \partial\delta\phi_+/\partial t &= -2\kappa\delta\phi_+ + 2\kappa\delta\phi_3 + f_+^0(t),\\ \partial\delta\phi_3/\partial t &= -g^2(I^0/\kappa)\delta\phi_+ - \kappa_3\delta\phi_3,\\ \partial\phi_-/\partial t &= f_-^0(t),\end{aligned} \tag{2}$$

where $I_1^0 = I_2^0 = I^0$ and the nonzero noise correlations are

$$\langle F_+^0(t) F_+^0(t')\rangle = -\langle F_-^0(t) F_-^0(t')\rangle = 4\kappa I^0 \delta(t-t')$$

and

$$\langle f_-^0(t) f_-^0(t')\rangle = -\langle f_+^0(t) f_+^0(t')\rangle = (\kappa/I^0)\delta(t-t').$$

The fluctuations $\delta I_\pm$, δI_3, $\delta\phi_+$, and $\delta\phi_3$ are damped with stable points of zero, while the decoupled signal-idler difference phase ϕ_- undergoes a continuous phase diffusion. The solution for ϕ_- is immediate. This phase diffusion with $\langle[\phi_-(t+\tau)-\phi_-(t)]^2\rangle = \kappa|\tau|/I^0$ was calculated by Graham and Haken,[6] who derived and solved equivalent operator equations to Eqs. (2). We have now calculated the full linearized spectrum for the long-time solutions for the stochastic phase and intensity variables ϕ_-, $\delta\phi_+$, $\delta\phi_3$, $\delta I_\pm$, and δI_3.

Stationary spectra for the field transmitted through a single-port cavity are defined:

$$S_{ij}(\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau}\langle \tilde{a}_i^\dagger(t)\tilde{a}_j(t+\tau)\rangle d\tau,$$
$$C_{ij}(\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau}\langle \tilde{a}_i(t)\tilde{a}_j(t+\tau)\rangle d\tau, \tag{3}$$

where $\tilde{a}$ denotes the field external to the cavity.[7] These are readily calculated in terms of our solutions to (2).[8] The elements $\langle a_i(t)\rangle$, $S_{12}(\omega)$, $C_{11}(\omega)$, ... are zero. The nonzero elements are

$$S_{11}(\omega) = S_{22}(\omega) = 2I^0L_0 - \tfrac{1}{8}L_{2\kappa} - \tfrac{1}{8}C_{\phi_+} + \tfrac{1}{8}C_{I_+},$$
$$C_{12}(\omega) = C_{12}^\dagger(\omega) = 2I^0L_0 + \tfrac{1}{8}L_{2\kappa} + \tfrac{1}{8}C_{\phi_+} + \tfrac{1}{8}C_{I_+}. \tag{4}$$

The solutions are linear combinations of the Lorentzian components associated with the variables ϕ_-, δI_-, $\delta\phi_+$, and δI_+, respectively, and are functions of the scaled parameters $\bar{\omega} = \omega/\kappa$, $r = \kappa_3/\kappa$, and $P = E/E_T$, where E_T is the threshold pump amplitude. The term $2I^0L_0 = \frac{1}{2}[(1/8I^0)^2 + \bar{\omega}^2]^{-1}$ is the large but narrow[9] Lorentzian due to the phase diffusion of ϕ_-. The remaining terms describe the small fluctuations due to the stable subset, which appear as small but broad components in the spectrum. The second term,

$$-\tfrac{1}{8}L_{2\kappa} = -\frac{1/8I^0+2}{4[(1/8I^0+2)^2+\bar{\omega}^2]},$$

describes fluctuations in the signal-idler intensity difference.[4] The third term describes fluctuations in the signal-idler phase sum and simplifies for large I^0 to

$$C_{\phi_+} = +\frac{4(r^2+\bar{\omega}^2)}{(2rP-\bar{\omega}^2)^2+\bar{\omega}^2(2+r)^2}.$$

The fourth term, describing the fluctuations in the signal-idler intensity sum, is positive and will be given elsewhere. The fluctuations in δI_- and $\delta\phi_+$ are negative in the P representation, implying noise levels below the vacuum or shot-noise limit.[4]

Consider the quadrature-phase amplitudes defined as $X_i^\theta(t) = e^{-i\theta}\tilde{a}_i + e^{i\theta}\tilde{a}_i^\dagger$ [we abbreviate $X_i^0(t) = X_i$ and $X_i^{\pi/2}(t) = Y_i$]. The average value $\langle X_i^\theta\rangle$ of any of the quadrature amplitudes is zero, because of the phase diffusion. The large phase fluctuations occur on a long time scale so that one may envisage measuring an instantaneous amplitude $X_1^\theta(t)$. The intensity undergoes small stable fluctuations on a much shorter time scale.

Although we cannot say *a priori* what the projection X_1 will be at a particular time, we note the signal and idler phases ϕ_1 and ϕ_2 are correlated, since $\phi_1+\phi_2$ has minimal fluctuations. The much smaller intensity fluctuations are *also* correlated, since $I_1 - I_2$ has minimal fluctuations. Thus we expect to infer quadrature-phase information of the signal by measuring the quadrature phase of the idler. The quantity $2V(\theta,\phi)$, where

$$V(\theta,\phi) = \tfrac{1}{2}\langle[X_1^\theta(t) - X_2^\phi(t)]^2\rangle$$
$$= 1 + 2[\langle a_1^\dagger a_1\rangle - \cos(\theta+\phi)\langle a_1 a_2\rangle],$$

is a direct measure of the error in our inferring of the signal amplitude $X_1^\theta(t)$, given an experimental determination $X_2^\phi(t)$ of idler amplitude. $V(\theta,\phi)$ is minimum for $\theta = -\phi$; thus a measurement of X_2 infers X_1, *and* Y_2 infers $-Y_1$. When $V(\theta,\phi) = 0$, there is a perfect correlation between $X_1^\theta(t)$ and $X_2^\phi(t)$.

We point out that the ability to infer "at a distance" either of two noncommuting signal observables, in this case X_1 and Y_1 for which $[X_1, Y_1] = 2i$, with a precision below the vacuum noise level on the signal is a direct example of the EPR paradox.[3] The minimum uncertainty relation for the signal conjugate variables is $\Delta X_1 \Delta Y_1 = 1$ and determines the vacuum noise level. Thus observations of $V(0,0) < 0.5$ *and* $V(\pi/2, -\pi/2) < 0.5$ constitutes an EPR experiment.

One may more easily measure experimentally[10] the

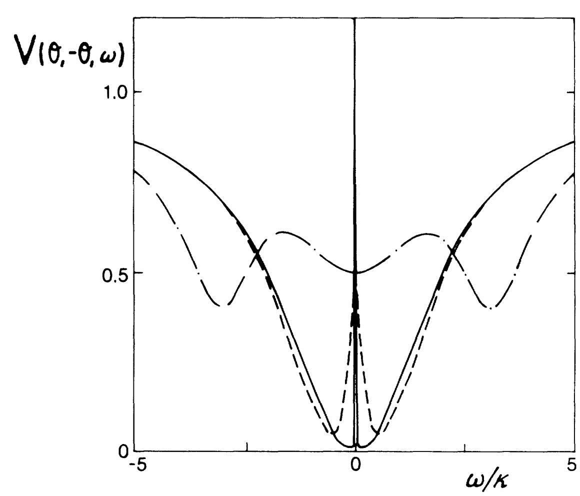

FIG. 1. Plot of $V(\theta,-\theta,\omega)$, the spectrum of fluctuation in the signal and idler quadrature amplitude difference $X_1^\theta - X_2^{-\theta}$: Solid line, near threshold. $E/E_T = 1.01$, $\kappa_3/\kappa = 0.01$, $I = 10$. Dash-dotted line, well above threshold with a good pump. $E/E_T = 50$, $\kappa_3/\kappa = 0.1$ ($I^0 > 10$). Dashed line, well above threshold with excellent pump. $E/E_T = 20$, $\kappa_3/\kappa = 0.01$ ($I^0 > 10$).

spectrum of fluctuations in the difference $X_1^\theta - X_2^\phi$ (Caves and Schumaker[11]),

$$V(\theta,\phi,\omega) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{i\omega\tau}\langle X_1^\theta(t) - X_2^\phi(t), X_1^\theta(t+\tau) - X_2^\phi(t+\tau)\rangle d\tau = 1 + 2\{S_{11}(\omega) - \cos(\theta+\phi)C_{12}(\omega)\}. \quad (5)$$

The expression with the optimal choice $\theta+\phi=0$ is $V(\theta,-\theta,\omega) = 1 - \frac{1}{2}L_{2\kappa} - \frac{1}{2}C_{\phi_+}$, determined only by the quantum fluctuations δI_- and $\delta\phi_+$. Maximum suppression of noise in δI_- (perfect signal-idler intensity correlation) corresponds to $L_{2\kappa}=1$. Maximum noise suppression in $\delta\phi_+$ (perfect correlation of signal-idler phase) corresponds to $C_{\phi_+}=1$. Since in an experimental situation $\theta+\phi$ is never quite optimal, a truer description of V above threshold is $V(\theta,\phi,\omega) = 1 + 2\{S_{11}(\omega) - jC_{12}(\omega)\}$. Here $j = \langle\cos 2\Delta(\theta+\phi)\rangle$ is the average over the jitter in local oscillator phases. The solution below threshold $(P<1)$ is

$$V(\theta,-\theta,\omega) = 1 - 4P/[(1+P)^2 \mp \omega^2],$$

a simple Lorentzian.[12]

Figure 1 shows V for various r, P, and fixed $j=0.99$. $V \rightarrow 0$ is indicative of an EPR correlation. The coherent-noise spike for $P>1$ at $\omega=0$ is due to phase diffusion. Near threshold, both phase $(\delta\phi_+)$ and intensity (δI_-) fluctuations are perfectly suppressed and $V(\theta,-\theta,\omega) \rightarrow 0$ near $\omega=0$. This is true in fact for all values of r and is also true below threshold as $P \rightarrow 1$. For very small r, nearly perfect suppression of phase fluctuations becomes possible well above threshold at higher frequencies $\bar{\omega} \sim (2rP)^{1/2}$. Figure 1, dash-dotted curve, illustrates the appearance of such side peaks. The central dip is the intensity fluctuation spectrum. Particularly interesting is the situation, depicted in the dashed curve, of an excellent pump $(r \gtrsim 0)$. The bandwidth for perfect intensity-fluctuation reduction extends out to $\bar{\omega} \sim 2$, and encompasses the frequencies $\bar{\omega} \sim (2rP)^{1/2}$ showing perfect phase-fluctuation reduction. Thus in summary we have perfect quantum correlation of signal-idler phase *and* intensity for fields of macroscopic photon number.

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