Are Attractors Relevant to Turbulence?

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The statistical hypothesis underlying the "strange attractor" explanation of fluid turbulence is suspect. Spatially extended systems generically exhibit long transients that preclude observation of the behavior governed by the asymptotic invariant measure. Even if the local dynamics is periodic, when it is coupled into a spatial system complex "turbulent" behavior can exist for times that grow faster than exponentially with increasing system volume. The nature of the attractor is irrelevant to the observed behavior when such systems are of even moderate size.

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The quarter-century old, and now conventional, answer to the title is affirmative. After a decade of active investigation, a number of experiments have implicated few-dimensional attractors in the production of complex, unpredictable "turbulent" behavior.[1] On the basis of these and other observations and analyses it has been concluded that the Lorenz-Ruelle-Takens hypothesis[2,3] of few-dimensional chaos is the proper explanation of the "nature of turbulence." A more circumspect summary of the contemporary picture is that chaos has been detected in "weak" turbulence, but that turbulence has not been explained in all its manifestations. The notion that an attractor, or invariant measure, underlies turbulent behavior dates back to Wiener.[4] Indeed, it formed the basis statistical *Ansatz* upon which his prediction theory and the ergodic theory of dynamical systems has been developed. Nonetheless, the assumption of a governing attractor is a significant statistical assumption[5] often left unstated in contemporary chaotic data analysis.

There are several alternative explanations of turbulence, however, that do not employ the *attractive hypothesis*[5]: spin-glass relaxation,[6-8] spatial noise amplification,[9-11] and, the subject of this Letter, transients. There are two broad classes of the latter: nonstationary and quasistationary transients. Two examples of the first are chaotic defect motion in lattice dynamical systems[12] and dislocations in video feedback.[13] With this type of transient behavior, the number of "defects" and the size of turbulent regions decreases with time via defect-defect annihilation, so that asymptotically the behavior simplifies. The relaxation time for disappearance of the complex patterns increases at most exponentially with system size.[14] That the initial behavior is in a transient regime is readily determined from time series from spatial probes. We call this nonstationary decay type-I transient turbulence (TT-I).

With quasistationary transients, the pattern evolution looks turbulent, statistical quantities appear to converge for all practical times, and these indicate very many-dimensional, unpredictable behavior. After a long time, however, the system suddenly falls onto a few-dimensional attractor. One cannot tell if the behavior is a transient, except via an observation interval that extends into the asymptotic regime. Furthermore, the attractor can be of such low dimensionality or dynamic simplicity that it indicates nothing of the structure of the many-dimensional subspace in which the transients move. We call this type-II transient turbulence (TT-II).

In an attempt to address problems such as these, we use a class of prototype spatially extended systems, that are discrete in space and time.[12,15,16] As a simple example of complexity arising in a spatially extended system that is *not* associated with an attractor, we introduce the piecewise-linear "dripping handrail" model,[12] given by

$$x_{n+1}^{i}=\frac{1}{2r+1}\sum_{j=-r}^{r} f(x_n^{i+j}),$$

with the local dynamics $f(x) \cong sx+\omega (\mathrm{mod}\, 1)$. Here $x_n^i \in [0,1)$ is the state at the ith site ($i=0,\ldots,N-1$) at time n, r the radius of the coupling, and N the number of sites. (We use nearest-neighbor coupling: $r=1$.) The local dynamics consists of an increase by roughly ω with each iteration and a sudden decrease in amplitude above a threshold $x_{\mathrm{drop}}=s^{-1}(1-\omega)$. This lattice is a grossly simplified model of a dripping fluid layer. The behavior of even an isolated drop is quite complex.[17] We note that $x_{n+1}=f(x_n)$ has been used as a model of single neuron dynamics[18] and of the stirred Belousov-Zhabotinsky chemical reaction.[19]

Here we treat the case $s \leq 1$ in order to study dynamics without local information production. The latter is guaranteed by a negative Lyapunov spectrum. The isolated map exhibits a stable period-25 orbit for $s=0.91$ and $\omega=0.1$. A typical evolution of the lattice from a random initial condition is shown with a space-time diagram in Fig. 1. Detailed investigation of this model indicates that the apparent complexity is due only to information mixing *in space.*[14] For this reason we refer to the behavior as *transient spatial chaos.*

To investigate the relative contribution of transients and attractors to the observed complexity, the dependence of the average transient length T_N and attractor period P_N on system size N was estimated from a Monte Carlo sampling of 10^3 random initial conditions. The calculations typically gave $<1\%$ error in T_N; P_N was measured exactly. Figure 2 demonstrates how T_N scales with system size. The data are well fitted by

$$T_N = T_1 \times 2^{[(N-1)/N_c]^\alpha},$$

with $N_c=21.5\pm0.5$, $T_1=149.5\pm0.5$, and $\alpha=3.0\pm0.2$. The estimated T_1 agrees with the average transient measured directly on an isolated map, $T_{f(x)}=149.8$. We have measured N_c, T_1, and α for a range of other parameter settings. The growth-controlling exponent α varied from about unity up to 3.2, indicating that the hyperexponential growth is robust. These studies also revealed that 90% of the total basin measure is accounted for by only four or fewer attractors, the number of observed attractors increases at most linearly and typically irregularly, and the increase of the average attractor's period is very slow (linear) compared with that of transients. We conclude that the dominant behavior for large system size is governed not by attractors with long periods but by extremely long transients.

During the TT-II transient epoch $t<T_N$ there is no indication of an orbit's ultimate fate. Indeed, most statistics appear to converge as if there were a well-defined measure on the transient subspace. To formalize this, we define the *quasistationary measure* $\mu_T(t,\mathbf{x})d^N x$ as the measure at state $\mathbf{x}$ that is invariant during some epoch T starting at time t. For $t>T_N$, it decays to the invariant measure associated with the attractor. A quantitative measure of stationarity is given by the temporal evolution of the Shannon entropy $H^k(t)$ of the k-site pattern distribution

$$P(t;x^{i+1},\ldots,x^{i+k}) = \int_{\{j=1,\ldots,i,i+k+1,\ldots,N\}} d\mu_T(t,x^j).$$

We approximate this by $P(s^k)$ where s^k is a length-k binary sequence obtained from the spatial pattern $(x^{i+1},x^{i+2},\ldots,x^{i+k})$: $(s^k)_i=1$, if $x_{\mathrm{drop}}<x^i\leq 1$, i.e., if water dropped from the site; $(s^k)_i=0$, otherwise. The temporal evolution of the spatial entropy shows a small fluctuation about a constant value during the transient epoch and then a sudden drop when the system falls into the (spatially simple) attractor (see Fig. 3).

We have also performed extensive investigations[14] using spatiotemporal power spectra, spatiotemporal mutual information or *coherence*,[14] and orbital recurrence.[20] All three analyses provide clear demonstrations of stationarity for $n<T_N$. In the time domain broad-band spectra and the exponential decay of coherence give

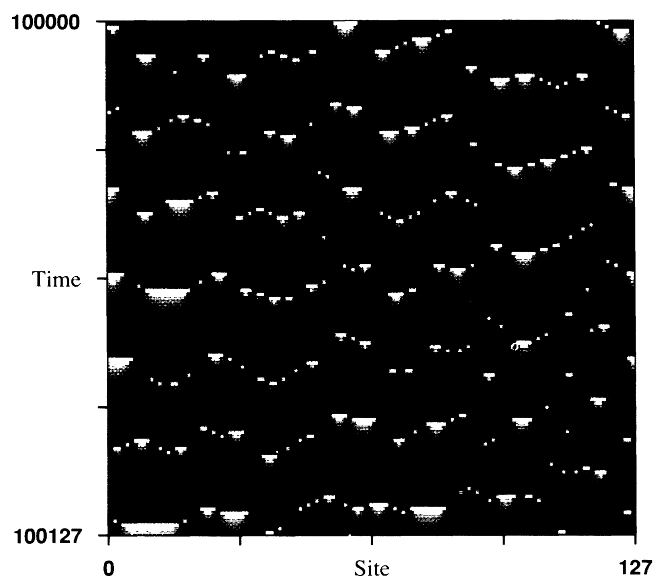

FIG. 1. Space-time diagram with site amplitudes $\{x_n^i\}$ in the range [0,1] plotted from black to white for a 128-site, spatially periodic lattice. 128 steps are shown, after 10^5 iterations of a random initial pattern.

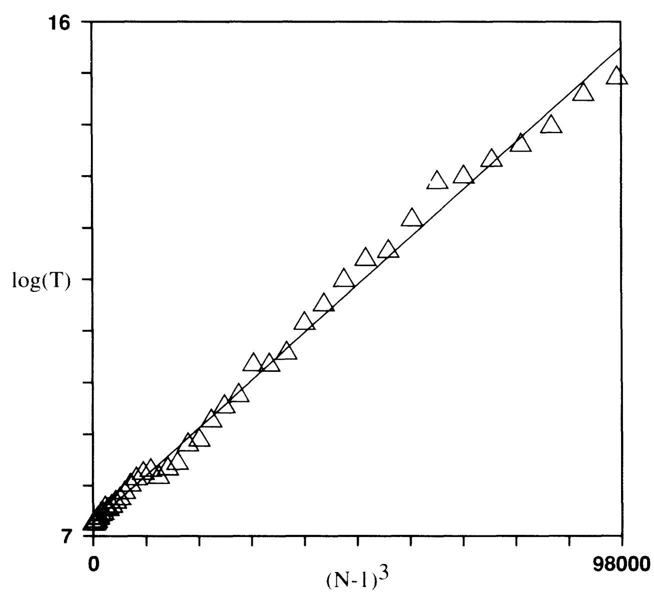

FIG. 2. Transient length T_N vs system size, $N=1,\ldots,47$. T_N varies from a low of $T_1\approx150$ to $T_{47}\approx6.17\times10^4$ iterations. The solid line shows the hyperexponential fit.

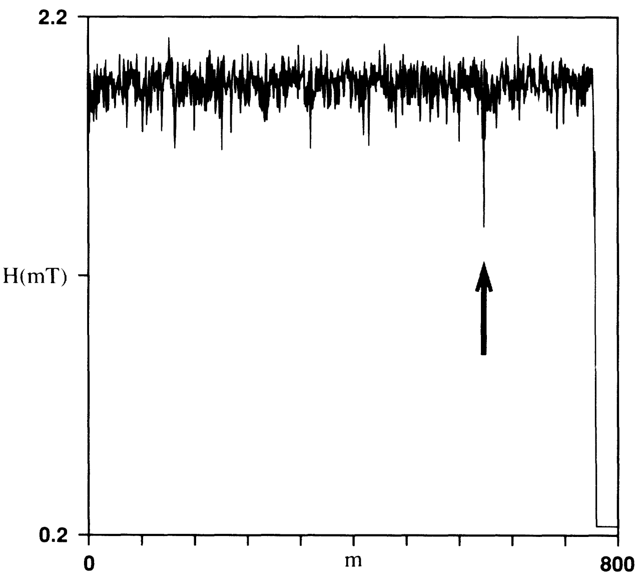

FIG. 3. Decay of the quasistationary measure to the invariant measure as revealed by the time evolution of the spatial entropy $H^k(t)$. The $k=10$ binary-sequence probabilities are estimated over $T=400$ time steps and over the entire lattice of $N=60$ sites. The arrow indicates close approach to the uniform attractor.

quantitative support to the observation that the transient behavior is complex and unpredictable. Additionally, the lack of f^{-1} scaling at low frequencies distinguishes the behavior from TT-I. In the space domain, spectra show a correlation length of $\approx N_c$ and a cascadelike decay of the form $k^{-\beta}$ with $\beta \approx 1.5$. A linear decay in the coherence, nonetheless, suggests relatively strong spatial correlation. Finally, orbital recurrence, which is particularly useful for investigation of many-dimensional state-space structures, reveals a convergence through a hierarchy of subspaces during the decay process.

TT-I and TT-II behavior reflect complex intrabasin and basin separatrix structures. Direct investigation of the underlying geometry is problematic in spatially extended systems given their high dimensionality. Nonetheless, an intrabasin structure organized as a many-dimensional maze explains much of the observed phenomena.[14] We picture the transient relaxation as a sequence of transitions through a hierarchy of *subbasins.* The subbasins are subspaces of a basin separated by walls through which an orbit cannot pass except at *portals.*

For TT-I the state space is organized as a hierarchy of subbasins of decreasing dimension. Patterns furthest away from the attractor are the most complex and move in correspondingly higher-dimensional subbasins. An orbit moves from one subbasin to another when two defects collide. This defines the *local* spatial conditions for the portal of some constant spatial length L determined only by the defect size and the *local* geometry of the annihilation process. The portal's state-space volume is then $V \approx c^L$ where c is a constant describing the average condition width at each site. Since the number of defects in a random initial pattern is proportional to N, the total probability P_N of making the necessary transitions down through the hierarchy is $P_N \approx c^{NL}$. If we assume ergodicity within each subbasin, the TT-I transient time $T_N \approx P_N^{-1}$ scales at most exponentially with system size.

Although the average complexity of the patterns is relatively constant during the TT-II transient epoch, they can make close approaches to uniformity. (See arrow in Fig. 3.) This suggests that for TT-II the subbasins consist of long tendrils that pass through the neighborhood of the simple attracting pattern. As if in a maze, the transient can be relatively close to the attractor at one time, but move very far away in order to find the correct path actually to reach the attractor. We assume that the complexity of the subbasin hierarchy is described by a direct product of the local basin structure of *each* site. The number of subbasins is proportional to N^γ, where γ describes the density of the tendrils. The lack of a localized annihilation mechanism indicates that the condition for passage through a portal is spatially global, depending on some fraction of N. The probability p of passing through a portal is then $p \approx c^{-N}$. The total probability is then the product of these over the hierarchy. The assumption of ergodicity at each stage of the relaxation yields a transient length $T_N \approx c^{N^{1+\gamma}}$ that grows hyperexponentially with system size and depends on the basin's tendril structure.

To appreciate the physical consequences of TT-II turbulence, consider the 128-site model of Fig. 1. This is a "small" system in comparison with the number of spatially distributed active modes in fully developed turbulent fluid flow, where the number of active modes is bounded by the number of dissipation-scale eddies contained in the flow. Nonetheless, if we assume that it is a physical model with a characteristic (iteration) time of 10^{-15} s, then an experimenter would have to wait more than 10^{40} yr to observe the ultimate periodic attractor, even in this mathematically ideal setting.

The nature of TT-II also leads to several computation theoretic consequences. First, the hyperexponential growth in T_N means that transient spatial chaos is, in principle, *unsimulatable* on finite-state machines for lattices larger than some size $N^* \approx (bN_c^\alpha)^{(\alpha-1)^{-1}}$, where b is the number of bits used to represent each local state. Thus, the periodic attractors in the lattice with $\alpha=3$ cannot be observed by IEEE double-precision simulation when the lattice contains more than approximately 800 sites. Second, in contrast to TT-I, the identification of the transient epoch has finite computational complexity in the probabilistic sense that, lacking prior knowledge of the asymptotic behavior, one can do no better than to make observations and wait for the decay.

We believe that the general result on the existence of dominating, arbitrarily long transients also applies to dynamical systems of other architectures, such as nonlinear neural networks, economic systems, evolutionary models, and generally those with many interacting subsystems. There is currently little theory describing transient behavior in such systems. In very few dimensions, existence of long transients has been theoretically studied only near critical points.[21] Automated experiments that mimic the finite-size scaling and that prepare the system in a specified ensemble of initial configurations appear to be the main avenues for observation of the rapid growth in transient epochs and so for detection of transient spatial chaos. Dynamical system methods need not be ignored, however, when TT-II is found. In the case of our estimating spatiotemporal equations of motion,[5] for example, transients are often desirable in order to increase the observed support of the dynamic and this, in turn, improves the estimates.

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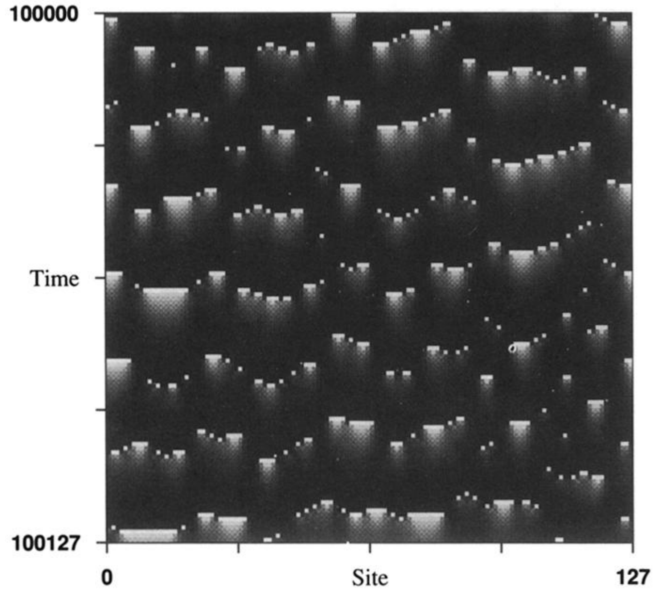

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