

Central Charge and Universal Combinations of Amplitudes in Two-Dimensional Theories Away from Criticality

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For a general isotropic two-dimensional theory near criticality, the universal singular free energy per correlation volume $f_s \xi^2$ is equal to $-(c/12\pi)(2-\alpha)(1-\alpha)^{-1}$, where c is the central charge of the theory at criticality, α is the usual specific-heat exponent, and ξ is defined (for $\alpha > 0$) in terms of the second moment of the energy correlations. Some generalizations of this result are also noted.

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In the past few years, our understanding of two-dimensional critical behavior has advanced considerably through the application of the principle of conformal invariance. However, as this principle only holds in the continuum limit exactly at the critical point, progress has been restricted to those properties of theories which may be related to their behavior at renormalization-group (RG) fixed points. A complete understanding of the universal properties of 2D critical systems should of course include the correlation functions away from criticality, in the scaling region where the correlation length, although finite, is much larger than any microscopic scale. At large distances, such systems are then equivalent to massive renormalizable quantum field theories.

As a first step in this program, one may try to calculate universal combinations of amplitudes of quantities which become singular as $T \rightarrow T_c$. These amplitudes generally are related to moments of the scaling forms of the correlation functions. One of the most fundamental of these universal numbers is the singular part of the free energy per correlation volume, $f_s \xi^2$. Here f_s is the singular part of the free energy per unit volume (measured in units of $k_B T_c$), which behaves like $A|t|^{2-\alpha}$, where as usual $t = (T - T_c)/T_c$, and ξ is the correlation length, which has¹ the asymptotic behavior $B|t|^{-\nu}$. Hyperscaling arguments,² supported by the RG,³ assert that the quantity $f_s \xi^2$ tends to a constant as $t \rightarrow 0$, and that the product AB is universal. The correlation length may be defined in many different ways, all suitable definitions giving the same exponent ν but different amplitudes B . One definition (although not the most practical one) is in terms of the second moment of the connected correlation function of the local energy density $\epsilon(r)$:

$$\xi^2 = \frac{\int r^2 \langle \epsilon(r) \epsilon(0) \rangle_c d^2 r}{\int \langle \epsilon(r) \epsilon(0) \rangle_c d^2 r}, \quad (1)$$

with use of a continuum notation. In the critical region, this correlation function has the scaling form $t^{-2+4\nu} \phi(rt^\nu)$. The denominator in (1) is just the specific heat per unit volume (measured in units of $k_B T_c$), whose

singular part is equal to $-\partial^2 f_s / \partial t^2 \sim -(2-\alpha)(1-\alpha) \times t^{-2} f_s$. Thus

$$f_s \xi^2 \sim -[(2-\alpha)(1-\alpha)]^{-1} t^2 \times \int r^2 \langle \epsilon(r) \epsilon(0) \rangle_c d^2 r, \quad (2)$$

which, from the scaling form above, is independent of t as $t \rightarrow 0$.

The main result of this Letter is that the integral on the right-hand side of (2) may be evaluated in terms of α and the conformal-anomaly number c of the critical theory. This is defined as follows. The stress tensor $T_{\mu\nu}$ is defined in terms of the response of the reduced Hamiltonian (or action) to an infinitesimal change of coordinates $x^\mu \rightarrow x^\mu + \alpha^\mu$, corresponding to a nonuniform RG transformation: $\delta \mathcal{H} = -(1/2\pi) \int \partial^\mu \alpha^\nu T_{\mu\nu} d^2 x$. It is convenient to use complex coordinates (z, z^*) . The components of the stress tensor are then given in terms of the Cartesian components by $T \equiv T_{zz} = \frac{1}{4}(T_{xx} - T_{yy} - 2iT_{xy})$, $\Theta \equiv 4T_{zz^*} = 4T_{z^*z} = T_{xx} + T_{yy}$, and $T^* \equiv T_{z^*z^*} = \frac{1}{4}(T_{xx} - T_{yy} + 2iT_{xy})$. These are respectively the spin 2, 0, and -2 components of the stress tensor, and Θ is its trace. At the critical point, $\Theta = 0$, and the conservation of $T_{\mu\nu}$ implies that $\partial_z T = \partial_z T^* = 0$. Rotational invariance then implies that the two-point function $\langle T(z) T(0) \rangle = (c/2) z^{-4}$, where c is a universal number, which then plays a fundamental role in the theory at the critical point. For example,⁴ the primary scaling operators correspond to highest-weight representations of the Virasoro algebra with central charge c , and the classification of such representations then leads to formulas for the critical exponents.⁵ Finite-size scaling amplitudes at criticality in certain geometries⁶ are also related to c .

Zamolodchikov⁷ has proved an important theorem whose significance extends to theories away from criticality: There exists an interpolating function C of the coupling constants of 2D quantum field theories which is nonincreasing along RG trajectories, and which, at an RG fixed point (FP), is equal to the value of c for that FP. It is now shown that the total change of C from one

FP to another is related to an integral like that in (2). Away from the FP, $\Theta \neq 0$. Rotational invariance constrains the two-point functions of T and Θ to have the forms

$$\begin{aligned} \langle T(z, z^*) T(0, 0) \rangle &= F(z z^*) / z^4, \\ \langle T(z, z^*) \Theta(0, 0) \rangle &= \langle \Theta(z, z^*) T(0, 0) \rangle \\ &= G(z z^*) / z^3 z^*, \\ \langle \Theta(z, z^*) \Theta(0, 0) \rangle_c &= H(z z^*) / z^2 z^{*2}. \end{aligned} \quad (3)$$

Conservation of the stress tensor $\partial_z \cdot T + \frac{1}{4} \partial_z \Theta = 0$ then implies that

$$\begin{aligned} \dot{F} + \frac{1}{4} (\dot{G} - 3G) &= 0, \\ \dot{G} - G + \frac{1}{4} (\dot{H} - 2H) &= 0, \end{aligned} \quad (4)$$

where $\dot{F} = z z^* F'(z z^*)$, etc. Eliminating G from the above, and defining $C = 2F - G - \frac{3}{8} H$, one obtains

$$\dot{C} = -\frac{3}{4} H. \quad (5)$$

Noting that $H > 0$, one sees that C is a nonincreasing function of $R = (z z^*)^{1/2}$. The RG then implies that C , evaluated at some fixed value of R , is decreasing along RG trajectories. Moreover, at a FP, $\Theta = 0$ and $C = 2F = c$. This is then Zamolodchikov's result.⁸

Returning to (5) one sees that, at some fixed value of the coupling constants, the total change in C from short to large distances is given by an integral of the two-point function of Θ . Now consider a noncritical theory with Hamiltonian⁹ $\mathcal{H} = \mathcal{H}^* + t \int \epsilon(r) d^2 r$, where \mathcal{H}^* is the FP Hamiltonian. Under a dilatation $r^\mu \rightarrow r'^\mu = (1 + \lambda) r^\mu$, with $\lambda \ll 1$, $\epsilon(r) \rightarrow (1 - \lambda x_\epsilon) \epsilon(r')$, where $x_\epsilon = 2 - \nu^{-1}$ is the scaling dimension of ϵ . Thus the total change in \mathcal{H} is $-\lambda(2 - x_\epsilon) t \int \epsilon(r) d^2 r$. However, by the definition of the stress tensor this change is $-(\lambda/2\pi) \int \Theta(r) d^2 r$. Thus

$$\Theta(r) = 2\pi t (2 - x_\epsilon) \epsilon(r). \quad (6)$$

With this kind of perturbation, the RG trajectory will flow from the critical FP with $C = c$ to a trivial high- (or low-) temperature FP with $C = 0$, i.e., the correlation functions will fall off exponentially fast at large distances. Thus, substituting (6) into (5) and noting that $\dot{C} = \frac{1}{2} R dC/dR$, we have

$$c = 6\pi^2 t^2 (2 - x_\epsilon)^2 \int_0^\infty R^3 \langle \epsilon(R) \epsilon(0) \rangle_c dR. \quad (7)$$

The integral is equal (apart from a factor of 2π) to that in (2). On expressing $2 - x_\epsilon$ as $2/(2 - a)$, one finds the main result quoted in the abstract.

The integrand in the denominator of (1) behaves like r^{-2x_ϵ} as $r \rightarrow 0$, and thus the integral converges only for $x_\epsilon < 1$ ($a > 0$). When $a \leq 0$, the integral must be cut off for $r < a$ (where a is a microscopic distance), and the denominator then behaves like $A_1 + A_2 |t|^{-a} \dots$, where A_1 and A_2 are constants with $A_1 \neq 0$. However,

the numerator remains ultraviolet finite. As a result, ξ as defined by (1) does not scale as $t^{-\nu}$ and $f_s \xi^2$ is no longer universal with this definition of ξ . Nevertheless, the "sum rule" of Eq. (7) remains valid. The only nontrivial model for which the energy-energy correlations are known exactly is the Ising model. For this case $a = 0$, and so the result must be verified at the level of Eq. (7). The reduced Hamiltonian¹⁰ is

$$\mathcal{H} = -K \sum_{x,y} \epsilon(x, y)$$

where

$$\epsilon(x, y) = s(x, y) [s(x + a, y) + s(x, y + a)],$$

and $t = K_c - K$. The energy correlations in the scaling region are¹¹

$$\langle \epsilon(R) \epsilon(0) \rangle_c = 4(\pi \xi_m)^2 [K_1^2(R/\xi_m) - K_0^2(R/\xi_m)], \quad (8)$$

where K_0 and K_1 are modified Bessel functions, and $\xi_m = (4|t|)^{-1}$ is the magnetic correlation length. On inserting (8) into (7) and performing the integrals, one finds $c = \frac{1}{2}$, as expected for the Ising model.⁴

Although the main result has been stated in terms of a thermal perturbation of the FP Hamiltonian, it has obvious extensions to other relevant scaling fields. For example, it applies to the case of a nonzero magnetic field, with γ replacing a , and ξ defined in terms of the second moment of the magnetic correlation function. A simple generalization applies to cases when the relevant perturbation causes crossover to a nontrivial FP. In that case, the left-hand side of (7) is replaced by the difference of the values of c at the two FP.

To conclude, a quantitative prediction of conformal invariance for noncritical systems has been given. In principle, it is amenable to experimental verification, in, for example, adsorbed systems. The version of the result applied to magnetic correlations, may, for example, be tested by a scattering experiment near a two-dimensional liquid-gas critical point.¹² When it is applied to the limit $n \rightarrow 0$ of the $O(n)$ model, the universal amplitude in the radius of gyration of ring polymers may be successfully predicted.¹³

It remains to be seen whether this is an isolated result, or the tip of an iceberg. The recent result of Zamolodchikov¹⁴ that for certain perturbations in some theories there exists an infinite number of conserved quantities suggests the second possibility.

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¹The amplitudes A and B are in general different for $T > T_c$.

and $T < T_c$. However, my result shows that the product AB is independent of the sign of $T - T_c$ in two dimensions, with my definition of ξ .

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⁸This version of the argument, which is slightly more transparent than that presented in Ref. 7, was presented by A. B. Zamolodchikov at the NORDITA-Chalmers University Seminar on Problems in Condensed Matter Physics, Göteborg, Oc-

tober, 1986 (unpublished).

⁹In my writing this in a continuum form, ϵ is assumed to be a renormalized operator in the critical point theory. That is, its two-point function is $\langle \epsilon(r)\epsilon(0) \rangle = r^{-2x_\epsilon}$, independent of any microscopic length scale a . t is then the renormalized coupling and it differs from the bare quantity $(T - T_c)/T_c$ by a factor of $a^{x_\epsilon - 2}$. It is important for the validity of the argument leading to Eq. (6) that the addition of a *relevant* perturbation to the Hamiltonian creates no new ultraviolet divergences. Thus the renormalized operator for $t \neq 0$ is the same as that in the critical theory.

¹⁰Here, both ϵ and t are bare quantities. However, only the product $t\epsilon$ enters into the result (7), and this is, by definition, the same for bare and renormalized quantities.

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