# Superconducting Ground State of Noninteracting Particles Obeying Fractional Statistics

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In a previous paper, Kalmeyer and Laughlin argued that the elementary excitations of the original Anderson resonating-valence-bond model might obey fractional statistics. In this paper, it is shown that an ideal gas of such particles is a new kind of superconductor.

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In a recent Letter,[1] Kalmeyer and I proposed that the ground state of the frustrated Heisenberg antiferromagnet in two dimensions and the fractional quantum Hall state for bosons might be the "same," in the sense that the two systems could be adiabatically evolved into one another without crossing a phase boundary. Whether or not this is the case is not presently clear. Indeed, the existence of a spin-liquid state of *any* spin-$\frac{1}{2}$ antiferromagnet in two dimensions has not been demonstrated. However, the case for a phase boundary's not being crossed is sufficiently strong that it is appropriate to ask what the consequences would be if this occurred. Adiabatic evolution is a particularly useful concept in the study of fractional quantum Hall "matter." So long as the energy gap remains intact, the "charge" of its fractionally charged excitations remains *exact* and the concomitant long-range forces between them, their fractional statistics, remain operative. This is why the fractional quantum Hall effect is so stable and reproducible. The persistence of the gap under evolution of the fractional quantum Hall problem into the magnet problem would allow us to make exact statements about the magnet without knowing *anything* about its Hamiltonian. In particular, the excitation spectrum of the magnet would be almost identical to that proposed by Kivelson, Rokhsar, and Sethna,[2] and completely within the spirit of the Anderson resonating-valence-bond idea,[3,4] except for one crucial detail: Both the chargeless spin-$\frac{1}{2}$ excitations, the "spinons," and the charged spinless excitations, the "holons," would obey $\frac{1}{2}$ fractional statistics.[5,6] The purpose of this Letter is to point out that this overlooked property may well account for high-temperature superconductivity.

Kalmeyer and I found the magnetic analog of the charge-$\frac{1}{3}$ quasiparticle of the fractional quantum Hall effect to be a spin-$\frac{1}{2}$ excitation, well described qualitatively as a spin-down electron on site $j$ surrounded by an otherwise featureless spin liquid. This particle is our version of the "spinon." Like the quasiparticle of the fractional quantum Hall state, it carries a "charge," that is, its spin, that is in a deep and fundamental sense fractional. In the limit that the antiferromagnetic interactions are turned off, the excitation spectrum of the magnet is purely bosonic. Spin-$\frac{1}{2}$ particles occur because these "elementary" excitations are fractionalized: Half the boson is deposited in the sample interior and half at the boundary. It was first pointed out by Halperin[6] that, in the fractional quantum Hall effect, the fractionalization of the electron charge $e$ into the quasiparticle charge $\frac{1}{3}e$ causes the quasiparticle to obey $\frac{1}{3}$ fractional statics. That is, each quasiparticle acts as though it were a boson carrying a magnetic solenoid containing magnetic flux $\frac{1}{3}\times hc/e$. This fact, deduced by Halperin from the experimentally observed fractional quantum Hall hierarchical states, was later shown by me[7] to follow from the analytic properties of the quasiparticle wave functions. It arises physically because the states available to the multiquasiparticle system must be enumerated differently from those available to fermions or bosons. In other words, it comes from counting. Now, it is clear by inspection that the preferred nature of this representation does not care about the existence of a lattice. Thus the validity of our identification clearly predicts that spinons obey $\frac{1}{2}$ statistics.

Let us now imagine doping this lattice with holes. The most natural way to do this, in my opinion, is first to make a spinon, thus fixing the spin on site $j$, and then remove the electron possessing that spin. It is necessary to make the spinon first because an electron cannot be removed before its spin state is known. If one simply rips an "up" electron from site $j$, one tacitly projects the ground state onto the set of states with the $j$th spin up, thus creating an excitation with spin 1. This may be thought of as a pair of spinons in close proximity. Unless the interaction between spinons is attractive and sufficiently large (Kalmeyer and I found it to be repulsive[1]), to make this "spin wave" will be more expensive energetically than to make an isolated spinon. Given that this occurs, the resulting spinless particle, the "holon," should also exhibit $\frac{1}{2}$ fractional statistics because it is a composite of a spinon and a fermion.

Assume now that we have a gas of such holons obeying fractional statistics. What are its properties expected to be? This question was addressed to some extent by Arovas *et al.*,[8] who computed the second virial coefficient of an ideal gas of particles obeying fractional statistics as a function of the fraction $\nu$. Not surprisingly, they found a smooth interpolation between the case of fermions, which acts like a classical gas with *repulsive* interactions, and that of bosons, which acts like a classical gas with *attractive* interactions. Thus, if we insist on thinking of these particles as fermions, we must conclude that there is an enormous attractive force between them. This is also evident when one considers the low-temperature properties. Fermions at density $\rho$ have a large degeneracy pressure, and thus a large internal energy, while bosons have neither. Since fractional-statistics particles are in between, they have, *vis-à-vis* fermions, attractive forces comparable in scale to the Fermi energy. It is also important that spinless particles obeying fractional statistics cannot undergo Bose condensation. They are not bosons. However, if the fraction is $\frac{1}{2}$, then *pairs* of particles are bosons.

There is therefore good reason to suspect that a gas of particles obeying $\frac{1}{2}$ statistics might actually be a superconductor with a charge-2 order parameter. Let us investigate this possibility by considering a gas of fractional-statistics particles described by the free-particle Hamiltonian

$$\mathcal{H}=\sum_{j}^{N}\frac{p_j^2}{2m}. \tag{1}$$

Any eigenstate of this Hamiltonian may be written in the manner

$$\Psi(z_1,\ldots,z_N)=\left[\prod_{j<k}^{N}\frac{(z_j-z_k)^{\nu}}{|z_j-z_k|^{\nu}}\right]\Phi(z_1,\ldots,z_N), \tag{2}$$

where $z_j$ denotes the position of the $j$th particle in the $x$-$y$ plane expressed as a complex number, $\nu=\frac{1}{2}$, and $\Phi$ is a Fermi wave function. This is the singular gauge transformation first discussed by Wilczek.[5] If we have an eigenstate $\Psi$ satisfying $\mathcal{H}\Psi=E\Psi$, then $\Phi$ satisfies $\mathcal{H}'\Phi=E\Phi$ where

$$\mathcal{H}'=\sum_{j}^{N}\frac{1}{2m}|\mathbf{p}_j+\mathbf{A}_j|^2, \tag{3}$$

and

$$\mathbf{A}_j(\mathbf{r}_j)=\nu\sum_{k\neq j}\hat{\mathbf{z}}\times\mathbf{r}_{jk}/|\mathbf{r}_{jk}|^2. \tag{4}$$

Thus, in the Fermi representation, each particle appears to carry a magnetic solenoid with it as it moves around in the sample. The vector potential felt by a particle is then the sum of the vector potentials generated by all the other particles. Because particles obeying $\frac{1}{2}$ statistics behave like fermions, in the sense that they possess degeneracy pressure, let us attempt to solve this problem in the Hartree-Fock approximation: We make a variational wave function that is a single Slater determinant constructed of orbitals $\phi_j(z)$ and minimize the expected energy. The orbitals then obey equations of the form

$$\mathcal{H}_{\mathrm{HF}}\phi_j(z)=\lambda_j\phi_j(z), \tag{5}$$

where $\mathcal{H}_{\mathrm{HF}}$ is the first variation of $\langle\mathcal{H}'\rangle$ and $\lambda_j$ is a Lagrange multiplier. The latter has the physical sense of a partial derivative of the total energy with respect to occupancy of the $j$th orbital. Since, in the mean-field sense, each particle must see a uniform density of magnetic solenoids carrying flux $\nu hc/e$, it is reasonable to guess the solution to be Landau levels, with the magnetic length $a_0$ related to the particle density $\rho$ by $a_0^2=(2\pi\nu\rho)^{-1}$. Self-consistency is achieved when the lowest $1/\nu$ Landau levels are filled. Thus, the fractions $\nu=1,\frac{1}{2},\frac{1}{3},\ldots$ are special cases in which a gap opens up in the fermionic spectrum.

Let us now test these equations in a case for which we know the answer, namely $\nu=1$, the noninteracting Bose gas. If the variational procedure describes this limit correctly, there is good reason to trust its predictions for $\nu=\frac{1}{2}$. Evaluating the self-consistent field with one Landau level filled, I obtain

$$\mathcal{H}_{\mathrm{HF}}=\tfrac{1}{2}E_0+(-E_0-\tfrac{1}{4})\Pi_0+\sum_{n\neq 0}\left[n+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left(\frac{1}{4}\sum_{l=1}^{k}\frac{1}{l}-\frac{1}{2k}\right)+\frac{1}{4(n+1)}\right]\Pi_n, \tag{6}$$

with

$$E_0=\int_0^{\infty}r^{-1}[1-e^{-r^2/2}]e^{-\alpha r}\,dr, \tag{7}$$

in units of the equivalent cyclotron frequency $\hbar\omega_c=2\pi\nu(\hbar^2/m)\rho$, where $\Pi_n$ denotes the projector onto the $n$th landau level, and $\alpha$ is a regulation parameter, effectively the inverse of the sample radius. Since $\mathcal{H}_{\mathrm{HF}}$ preserves Landau-level index, the state we guessed is a true variational minimum. Note, however, the logarithmic divergence in the Lagrange-multiplier spectrum, implying that the cost to inject either a "particle" or an "antiparticle" is arbitrarily large. This is absolutely the correct result. The noninteracting Bose gas has no low-lying fermionic excitations. The fact that these divergences are logarithmic suggests that the relevant excitations are actually quantum vortices. That this is, in fact, the case may be seen by our imagining an extra particle to be placed at the origin and calculating the expected current density $\langle\mathbf{J}(r)\rangle$. The current-density operator may be written $\mathbf{J}(r)=m^{-1}(\mathbf{p}+\mathbf{A}_{\mathrm{old}}+\Delta\mathbf{A})$, where $\mathbf{A}_{\mathrm{old}}$ is the vector potential in the absence of the extra particle and $\Delta\mathbf{A}$ is the vector potential generated by a solenoid at the origin. Since $\langle \mathbf{p}+\mathbf{A}_{\text{old}}\rangle=0$, the current density must just be the particle density at $r$ times $\Delta\mathbf{A}$, or a vortex of magnetic strength $hc/e$.

The expected energy of the ground state state is $N/4$ in these units. This is considerably higher than the correct answer of zero. This discrepancy is due to the fact that the wave function is forced by its construction to go to zero when the particles come together. It is thus more appropriate for the description of real helium than noninteracting bosons. It should also be noted that this behavior is actually required of the $\nu=\frac{1}{2}$ wave function. Let us observe finally that the broken symmetry characteristic of a superfluid is not expressly exhibited by the Hartree-Fock ground state. This is as expected. Is was shown by Bogoliubov[9] that the broken symmetry of a Bose gas is absent unless the bosons interact. All that is required for the symmetry to break is a weak interparticle repulsion and the presence in the "unperturbed" Bose gas of a collective mode dispersing quadratically with the mass of the bare particles. In the present case, it is easy to see that the variational solution possesses a collective mode that disperses quadratically. Since $\langle \mathcal{H}'\rangle/N$ is proportional to the particle density, the pressure is constant, and thus the bulk modulus is zero. It is a straightforward matter to calculate the mass of this mode by the magnetoexciton procedure of Kallin and Halperin.[10] My preliminary results give a value of approximately $\frac{1}{2}$ the bare mass. The precise value of this mass is not so important as the fact that it is of order unity. The collective mode may be thought of both as a density wave and as a magnetoexciton consisting of a hole in the lowest Landau level and a particle in the first excited Landau level, bound together by a logarithmic potential

Let us now turn to the case of interest, $\nu=\frac{1}{2}$. It is so similar to the $\nu=1$ case that there is little to say. Assuming two Landau levels filled, I obtain

$$\mathcal{H}_{\text{HF}}=\tfrac{11}{16}+\tfrac{1}{4}E_0+(-\tfrac{1}{2}E_0-\tfrac{1}{8})\Pi_0+(-\tfrac{1}{2}E_0+\tfrac{29}{24})\Pi_1 + \sum_{N\geq 2}\left[n+\sum_{k=1}^{n}\binom{n}{k}(-1)^k\left(\frac{1-k}{4}\sum_{l=1}^{k}\frac{1}{l}-\frac{1}{4k}\right)+\frac{3}{8(n+1)}-\frac{1}{8(n+2)}\right]\Pi_n. \quad (8)$$

Thus, we again have a true variational solution with vortexlike fermionic excitations. Repeating the arguments for $\nu=1$, I find that the flux quantum to which the vortices correspond is $hc/2e$, exactly as expected of a charge-2 superfluid. Once again, a soft collective mode will mix into the ground state to break the symmetry when repulsive interactions are introduced. Thus, the ground state is a superfluid very similar to liquid helium except that the charge of its order parameter is 2.

While considerable work needs to be done to quantify this picture, some of its implications may be seen at a glance. By far the most important is that a normal-metal state, in the sense of Fermi-liquid theory, does not exist, just as Anderson[4] suggested. A corollary is that the occurrence of superconductivity does not have anything to do with self-consistent opening of an energy gap in the tunneling spectrum, as occurs in the BCS theory. Indeed, I find that tunneling cannot even be understood outside the context of the creation of spinons by the tunneling event. It should be noted that this is also consistent with Anderson's views.[11] A critical prediction is that an energy gap *must* occur in the spin-wave spectrum, the spin analog of the collective mode[12] of the fractional quantum Hall state. This is because the presence or absence of this gap is precisely the difference between the disordered and ordered states.

In summary, it is possible that high-$T_c$ superconductivity can be accounted for by the following simple idea: The force mediated by the spins of the Mott insulator is not an attractive potential, but rather an attractive *vector* potential.

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