Fluctuations of Solid Membranes

Joseph A. Aronovitz and T. C. Lubensky
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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The free energy of a D-dimensional elastic solid embedded in a d-dimensional space and subject to an extrinsic bending energy is defined, and the solid's flat phase studied. An $\epsilon=4-D$ expansion about the upper critical dimension $D_{uc}=4$ is performed; exponents characterizing the renormalization of the Lamé coefficients λ and μ, and the rigidity κ, are computed to $O(\epsilon)$; and exact equations connecting these exponents are derived. Near $D=4$, fluctuations increase the rigidity, and so tend to stabilize the flat phase.

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It has long been known that in thermodynamic equilibrium, long-chain polymers have a finite persistence length beyond which they are orientationally decorrelated.[1] One thinks of a polymer in a good solvent as a one-dimensional chain whose conformations fluctuate in a d-dimensional space. Recently, a large theoretical effort has been devoted to the understanding of the statistical mechanics of D-dimensional surfaces whose conformations fluctuate in the d-dimensional space $\mathbb{R}^d$.[2] Often, D is taken to be 2 or $d-1$. It is now well established that, like a polymer, a fluid membrane without surface tension whose energetics are dominated by a curvature energy of strength (rigidity) κ has a finite, rather than infinite, persistence length for $T>0$.[3-5] Very different results, however, have been recently obtained for the case of membranes with fixed connectivity, or with internal crystalline order.

Nelson and Peliti[6] have analyzed the statistical mechanics of an almost flat two-dimensional elastic membrane embedded in three-space, subject to the bending energy described by Helfrich.[7] They treat the phonon modes in a Gaussian approximation and show that, in a self-consistent theory which assumes finite renormalization of elastic constants, a phonon-mediated long-range interaction between capillary waves leads to a renormalized rigidity of the form $\kappa(q)\sim Cq^{-\eta}$ with $\eta=1$. They further argue that any positive value of η will stabilize a low-temperature flat phase. Toner has pointed out that curvature fluctuations renormalize elastic constants downward.[8] Unless this renormalization of the elastic constants of the full nonlinear theory is strong enough to reduce η from 1 to 0, one expects a flat phase to exist at sufficiently low, but finite, temperature.

This conclusion is reinforced in a recent investigation by David and Guitter of a two-dimensional elastic solid with rigidity κ and infinite elastic constants, which is embedded in $\mathbb{R}^d$.[9] They show in a $1/d$ expansion that this unstretchable membrane displays a finite-κ crumpling transition between a flat phase with a Hausdorff dimension of two and a crumpled phase with infinite Hausdorff dimension. A crumpling transition, in a model with finite elastic constants, between a flat solid phase and a crumpled tethered surface has also recently been investigated via the ϵ expansion by Paczuski, Kardar, and Nelson.[10]

Further evidence of the existence of the flat phase has been provided by Kantor and Nelson.[11] They have used Monte Carlo techniques on a tethered-surface model with $D=2$, $d=3$, and an energy of the form

$$E=-\kappa\sum_{\langle\alpha,\beta\rangle}(\hat{\mathbf{n}}_\alpha\cdot\hat{\mathbf{n}}_\beta-1)+\sum_{\langle i,j\rangle}V(\mathbf{r}_i-\mathbf{r}_j).$$

In this equation, κ is proportional to the continuum rigidity, $\hat{\mathbf{n}}_\alpha\in\mathbb{R}^d$ is normal to the αth plaquette, and $V(r)$ is a nearest-neighbor tethering interaction. These authors find evidence of a continuous, finite-κ crumpling transition between low-temperature (large κ) flat and high-temperature crumpled phases, within the size limits of their calculation.

The purported existence of a flat membrane phase naturally leads one to consider the general question: How do thermal fluctuations modify the elasticity of a D-dimensional solid which is embedded in a d-dimensional space (with $D<d$) and is given a reasonable energetic cost for extrinsic curvature? In this Letter, we investigate such a surface. We find that below the upper critical dimension $D_{uc}=4$, conventional elasticity theory breaks down and all elastic constants are infinitely renormalized in the infrared. We investigate the effects of thermal fluctuations in this regime using an $\epsilon=4-D$ expansion, derive exact relations between exponents, compute these exponents to $O(\epsilon)$, and discuss our results' implications for the theory's lower critical dimension.

For our investigation, we first must construct a long-wavelength free-energy functional appropriate to a homogeneous and isotropic D-dimensional solid (membrane) whose mass points are indexed by a vector $\mathbf{x}\in\mathbb{R}^D$. Let the position in $\mathbb{R}^d$ of the mass point indexed be $\mathbf{x}$ by the vector $\mathbf{r}(\mathbf{x})$, and assume that the configuration $\mathbf{r}(\mathbf{x})=(\mathbf{x},\tilde{0})$ is one of minimum energy. We here use $\tilde{0}$ to denote the zero vector in $\mathbb{R}^{d_c}$, and denote the solid's codimension $d-D$ by d_c. By continuum elastic theory,[12] the associated stretching energy is a scalar function of the strain tensor $u_{ij}\equiv\frac{1}{2}(g_{ij}-\delta_{ij})=\frac{1}{2}(\partial_i\mathbf{r}$

$\cdot \partial_j \mathbf{r} - \delta_{ij}$), where $\partial_i \equiv \partial/\partial x_i$. Expansion in u_{ij} to leading order, after demanding rotational invariance in both $\mathbb{R}^D$ and $\mathbb{R}^d$, yields the standard stretching free energy,

$$F_s = \tfrac{1}{2}\int d^D x(\lambda u_{kk}^2 + 2\mu u_{ij}^2),$$

where λ and μ are Lamé coefficients. One can show that, for D near 4, higher-order invariant combinations of u_{ij} are irrelevant operators.

The stretching energy F_s is exclusively a function of the metric tensor g_{ij}, and thus is a geometrically intrinsic quantity. In other words, F_s is independent of the solid's embedding in $\mathbb{R}^d$. To stabilize the solid with respect to fluctuations in the d_c directions perpendicular to itself, we must add to the free energy an extrinsic term; one which explicitly depends on the embedding of the surface. We expect such a term to correspond to a coarse-grained bond-bending energy. Accordingly, we consider a bending energy F_b which is a functional both of the extrinsic curvatures measured by the second fundamental form $\mathbf{K}_{ij}$ and of the intrinsic geometry characterized either by the metric tensor g_{ij} or equivalently, by the strain tensor.

To derive a form for $F_b[\mathbf{K}, u_{ij}]$ which will be valid for arbitrary D and d, we first consider the special case of $d_c = 1$ so that the surface has a single normal vector field $\hat{\mathbf{n}}(\mathbf{x})$. For this case we choose F_b to be the continuum limit of Kantor and Nelson's bending energy[11]

$$F_b = \lim_{\text{continuum}} \tilde{\kappa} \sum_{\langle \alpha,\beta \rangle} (1 - \hat{\mathbf{n}}_\alpha \cdot \hat{\mathbf{n}}_\beta)$$

$$= \lim \tfrac{1}{2}\tilde{\kappa} \sum (\hat{\mathbf{n}}_\alpha - \hat{\mathbf{n}}_\beta)^2 = \tfrac{1}{2}\kappa \int d^D x (\nabla \hat{\mathbf{n}})^2. \quad (1)$$

One can show that $(\nabla \hat{\mathbf{n}})^2 = \mathbf{K}_j^i \cdot \mathbf{K}_{ij}$, where the second fundamental form $\mathbf{K}_{ij}$ is the vector of tensors $\mathbf{K}_{ij} = \partial_i \partial_j \mathbf{r} - \Gamma_{ij}^k \partial_k \mathbf{r}$. Thus we can express F_b in terms of the extrinsic curvature:

$$F_b = \tfrac{1}{2}\kappa \int d^D x \, \mathbf{K}_j^i \cdot \mathbf{K}_{ij}, \quad (2)$$

and can analytically continue in d by choosing (2) as the general form for F_b. It is worth noting that when $D = 2$ and $d = 3$, (2) is consistent with the elasticity theory of thin plates conventionally known as shell theory.[13]

Our expression for F_b should be contrasted with the bending free energy F_H which Helfrich used to describe a fluid membrane.[7] Up to total derivatives, one can write

$$F_H = \tfrac{1}{2}\kappa \int dS H^2 = \tfrac{1}{2}\kappa \int d^D x \sqrt{g} \mathbf{K}_j^i \cdot \mathbf{K}_i^j.$$

Near $D = 4$, one can show that both free energies are identical up to irrelevant operators; however, in general $F_b \neq F_H$. That the bending energy associated with a solid is different from that of a fluid membrane should not be too surprising. A mathematical description of a constant-density fluid surface cannot depend on the coordinates used. In contrast, the solid's local density is a function of the displacements. Because $\mathbf{x}$ indexes mass points, the solid has a special coordinate system—the one in which the mass density $\rho(\mathbf{x})$ is constant. The solid does not have the fluid's coordinate transformation invariance because any stretching induces microscopic bond movement and thus induces changes in the bending energy, whether or not the transformation changes the embedded geometry.

We are interested in investigating the surface in its low-temperature flat phase. Accordingly, it is sensible to assume that displacements are small, and that self-overlaps will be negligible. We represent the position vector $\mathbf{r}(\mathbf{x})$ in the form

$$\mathbf{r} = (\mathbf{x} + \mathbf{u}, \tilde{h}); \quad \mathbf{x}, \mathbf{u}(\mathbf{x}) \in \mathbb{R}^D; \quad \tilde{h}(\mathbf{x}) \in \mathbb{R}^{d_c};$$

and average over both the phonons $\mathbf{u}(\mathbf{x})$ and the capillary waves $\tilde{h}(\mathbf{x})$. Our full phenomenological free energy F is the sum of F_b and F_s. After expansion of F in $\mathbf{u}$ and $\tilde{h}$, an analysis of the various terms' engineering dimensions[14] shows that F has upper critical dimension $D_{uc} = 4$. Near four dimensions F can be expanded in the form

$$F = \tfrac{1}{2}\int d^D x [\kappa(\nabla^2 \tilde{h})^2 + 2\mu \bar{u}_{ij}^2 + \lambda \bar{u}_{ii}^2] + \text{irrelevant}, \quad (3)$$

where $\bar{u}_{ij} \equiv \partial_i u_j + \partial_j u_i + \partial_i \tilde{h} \, \partial_j \tilde{h}$ is the relevant part of the strain tensor. One should notice that F_b's sole relevant contribution to F is the Gaussian term $(\kappa/2) \times (\nabla^2 \tilde{h})^2$. All relevant nonlinearities originate from F_s.

One can show by rescaling $\mathbf{u}$ and $\tilde{h}$ that perturbation theory in the nonlinear terms actually corresponds to an expansion in the two coupling constants $\bar{\lambda} \equiv \lambda/\kappa^2$ and $\bar{\mu} \equiv \mu/\kappa^2$. Accordingly, the flat (high κ) phase occurs at small values of the coupling constants and can be studied with the aid of a perturbative renormalization-group analysis. In particular, the Gaussian ($\bar{\lambda} = \bar{\mu} = 0$) theory corresponds to $\kappa = \infty$, i.e., the flat phase at $T = 0$.

The Gaussian part of F can be diagonalized by our writing $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$, where $\mathbf{u}_{\perp(\parallel)}$ is the transverse (longitudinal) part of $\mathbf{u}$. In the limit $\mu \to 0$, $\tilde{h}$ is only coupled to $\mathbf{u}_\parallel$. Accordingly, we can define a model for a shearless elastic medium, e.g., a "fixed connectivity fluid" by setting μ equal to zero and suppressing the transverse phonons (i.e., by setting $\mathbf{u}_\perp = 0$). In this limit the rigidity is not renormalized, because $\lambda u_{ii}^2 = \lambda[\nabla \cdot \mathbf{u} + (\nabla \tilde{h})^2/2]^2$, so that $u_\parallel$'s coupling to $\tilde{h}$ can be removed by a shift of $u_\parallel$. Integrating $\mathbf{u}$ out[6] thus yields a Gaussian effective theory for $\tilde{h}$ with an unchanged value for κ.

The presence of transverse phonons, along with nonzero μ, makes the theory nontrivial. We have analyzed (3)'s infrared behavior using the field-theoretical renormalization group[14] in a $D = 4 - \epsilon$ expansion at fixed codimension d_c after renormalizing F by minimal subtraction. As for the case of fluid surface models,[5,15] it proved useful in renormalizing to take advantage of the

TABLE I. η functions, fixed-point values of $\hat{\lambda}$ and $\hat{\mu}$, and exponents at the fixed points. (For clarity of presentation, we have written $\hat{\lambda}\equiv\lambda$ and $\hat{\mu}\equiv\mu$ in the η functions.) In all cases, the equations $\mu(-\epsilon+\eta_\perp+2\eta)=(\lambda+2\mu)(-\epsilon+\eta_\parallel+2\eta)=0$ are satisfied.

Fixed point	$\hat{\mu}^*$	$\hat{\lambda}^*$	$\eta=\frac{5\mu(\mu+\lambda)/2}{\lambda+2\mu}$	$\eta_\perp=\frac{1}{12}\mu d_c$	$\eta_\parallel=\frac{\mu^2+2\lambda(\lambda+\mu)}{\lambda+2\mu}\frac{d_c}{4}$
1	0	0	0	0	0
2	0	$2\epsilon/d_c$	0	0	ϵ
3	$\frac{12\epsilon}{20+d_c}$	$\frac{-6\epsilon}{20+d_c}$	$\frac{\epsilon}{2+d_c/10}$	$\frac{\epsilon}{1+20/d_c}$	$\frac{\epsilon}{1+20/d_c}$
4	$\frac{12\epsilon}{24+d_c}$	$\frac{-4\epsilon}{24+d_c}$	$\frac{\epsilon}{2+d_c/12}$	$\frac{\epsilon}{1+24/d_c}$	$\frac{\epsilon}{1+24/d_c}$

original theory's invariance under rotations and translations in $\mathbb{R}^d$. These continuous symmetries lead to Ward identities from which one can prove that the entire theory will be regularized if one chooses renormalized elastic constants κ_R, λ_R, and μ_R which make finite the momentum-space vertex functions $[d/d(q^2)]\Gamma^R_{uu}$ and $[d^2/d(q^2)^2]\Gamma^R_{hh}(q)$.

After renormalizing at a length scale l, we derive recursion relations for the dimensionless renormalized coupling constants $\hat{\mu}\equiv\bar{\mu}_R l^\epsilon$ and $\hat{\lambda}\equiv\bar{\lambda}_R l^\epsilon$ by demanding that bare correlation functions be independent of l. To $O(\epsilon)$ the associated β functions are

$$\beta_{\hat{\mu}}=[-4\epsilon\hat{\mu}+\hat{\mu}^2(\tfrac{1}{3}d_c+20A)]/16,$$

$$\beta_{\hat{\lambda}}=[-4\epsilon\hat{\lambda}+\hat{\mu}^2\tfrac{1}{3}d_c+2\hat{\lambda}^2 d_c+2\hat{\lambda}\hat{\mu}(d_c+10A)]/16,$$

where $A=(\hat{\lambda}+\hat{\mu})/(\hat{\lambda}+2\hat{\mu})$. Solving the fixed-point equations $\beta_{\hat{\lambda}}=\beta_{\hat{\mu}}=0$ leads to the four solutions shown in Table I. The flow diagram is shown in Fig. 1. For $D\geq 4$, the Gaussian fixed point is stable, and the nonlinear terms are irrelevant. Below four dimensions, any thermodynamically stable theory will be driven to fixed point (4) at long wavelengths. The boundary lines $\hat{\mu}=0$ and $\hat{\lambda}=-\hat{\mu}/2$ correspond [to $O(\epsilon)$] to the limits of stability of the D-dimensional solid—which by itself is thermodynamically stable only when both μ and the bulk modulus $B=\lambda+2\mu/D$ are positive.[12] The line $\mu=0$ corresponds to the fixed-connectivity fluid, which is driven to fixed point (2) at long wavelengths. Fixed point (4) is globally attractive and controls the physics of the flat phase. The fixed point controlling the crumpling transition[9,10] occurs at coupling constants $\hat{\lambda}$, $\hat{\mu}$ of order unity near $D=4$, and is thus beyond the reach of our low-temperature renormalization group. Our theory is thus analogous to that of the elasticity of smectic liquid crystals for $d\leq 3$,[16] but not analogous to systems such as the nonlinear σ model, where low-temperature renormalization groups at dimensions just above the lower critical dimension give information about disordering phase transitions.

We can, however, compute the effects in the flat phase of the d_c extra dimensions on the solid's elastic constants. At fixed points (2) and (4), the vertex functions pick up anomalous dimensions so that the elastic constants are nontrivially renormalized. The propagators obey scaling laws characterized by the exponents η, $\eta_\perp$, and $\eta_\parallel$ defined by

$$\Gamma_{u_\perp u_\perp}(q)\sim C_\perp q^{2+\eta_\perp};\quad \Gamma_{u_\parallel u_\parallel}(q)\sim C_\parallel q^{2+\eta_\parallel};$$

$$\Gamma_{hh}(q)\equiv q^4\kappa(q)\sim C_\kappa q^{4-\eta}.$$

The exponents are computed by the evaluation of η functions at a given fixed point. We have tabulated the fixed points, η functions, and critical exponents at each fixed

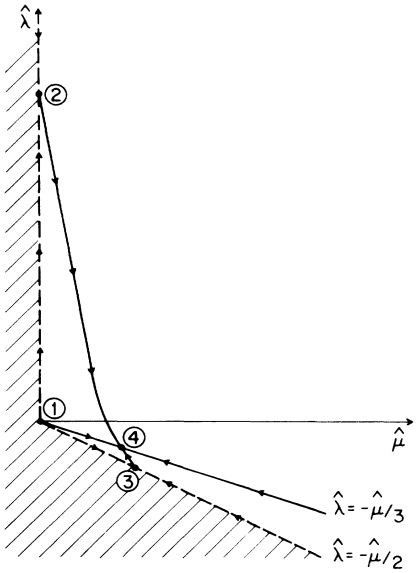

FIG. 1. Flow diagram in the $(\hat{\mu},\hat{\lambda})$ plane for $d_c=1$. The $O(\epsilon)$ flows along the eigendirections at each fixed point are schematically drawn in. The lines $\hat{\mu}=0$, $\hat{\lambda}=-\hat{\mu}/2$, and $\hat{\lambda}=-\hat{\mu}/3$ are invariant submanifolds under renormalization-group flows to $O(\epsilon)$. The D-dimensional solid is, by itself, unstable in the shaded region. The fixed-connectivity fluid corresponds to the line $\hat{\mu}=0$. For $\epsilon\geq 0$ ($D\leq 4$), fixed point (4) is globally stable, and fixed point (2) is stable along the $\hat{\mu}=0$ axis. For negative ϵ, the Gaussian fixed point (1) is globally stable.

point in Table I. The theory's underlying rotation invariance leads to connections between η and β functions. In particular, we can prove that at any fixed point the equations

$$\mu(-\epsilon+\eta_\perp+2\eta)=0, \quad (\lambda+2\mu)(-\epsilon+\eta_\parallel+2\eta)=0,$$

are *exactly* satisfied. These equations imply that $\eta_\parallel=\epsilon$ is true to all orders in ϵ for the fixed-connectivity fluid (for which $\eta\equiv 0$), and that

$$\eta_\perp=\eta_\parallel\equiv\eta_u=4-D-2\eta \tag{4}$$

for the solid. Table I is consistent with these results. The renormalization of elastic constants described by the above equation is similar to that occurring in smectic liquid crystals in $d\leq 3$.[16]

As stated above, our calculation presumes that the surface remains asymptotically flat and exhibits long-range order in its tangent vectors. Nelson and Peliti[6] have shown that such long-range order exists if the correlation function $\langle(\nabla h_a)^2\rangle$ stays finite. This condition, which also implies that the surface's Hausdorff dimension[5] $D_{\rm H}$ is equal to D, is satisfied whenever

$$D-2+\eta>0. \tag{5}$$

Thus, the bending terms in F act to stabilize the flat phase whenever η is positive, which it is near $D=4$. If η remains positive down to $D=2$, the rigidity will extend downward the range of D for which the flat phase exists to a lower critical dimension $D_{\rm lc}$ satisfying $2>D_{\rm lc}>1$. From (5) we see that at $D_{\rm lc}$, $\eta=2-D_{\rm lc}$, which, in conjunction with (4), implies that $\eta_\perp=\eta_\parallel=D_{\rm lc}$ at $D_{\rm lc}$. Insertion of these values into the exponents' defining relations shows that at $D_{\rm lc}$, $\Gamma_{uu}\sim\Gamma_{hh}\sim q^{2+D_{\rm lc}}$. The spectra of the phonons and of the capillary waves scale identically, and the surface crumples.

It is interesting to observe that there is a second critical dimension D_h defined by $\langle u^2\rangle=\infty$, or by $D_h-2-\eta_u=0$. Below D_h, long-range periodic order ceases to exist. If η_u remains positive down to $D=2$, then $D=2$ is less than D_h and a two-dimensional solid membrane would have bond-angle but not positional long-range order. It would thus be a fixed-connectivity hexatic.

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