Quantum Noise in Phase Conjugation

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We show by means of a quantum electrodynamic calculation that excess noise is inherent in the process of optical phase conjugation, both in that the state of the field leaving a phase-conjugate mirror can always be described classically (i.e., its phase-space density is positive semidefinite) and in that the fluctuations in the generated field are always greater than those predicted by Poisson statistics. Except in special cases, quantum noise imposes a limitation on the ability of phase conjugation to remove the effects of aberrations.

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Optical phase conjugation is a technique that has proved effective for the elimination of aberrations imparted on an optical wave front.[1,2] However, there are certain processes that can limit the performance of phase-conjugate mirrors (PCM's). For example, Agarwal, Friberg, and Wolf[3] have shown that a limitation to the aberration-correcting ability of phase conjugation exists even for the case of a PCM that is ideal in the sense that it produces no noise and returns precisely the phase conjugate of the incident field. Specifically, these authors show that for an aberrator in the form of a lossless scatterer the incident wave front will be reconstructed perfectly only if the aberrator produces no backscattering or if the phase-conjugate reflectivity is equal to unity. The performance of PCM's can also be degraded by fluctuations of the phase-conjugate signal. These fluctuations can result from instabilities in the specific nonlinear optical interaction used to produce the phase-conjugate signal[4] or can result from noise intrinsic to the phase-conjugation process.

In this Letter, we show by means of a quantum electrodynamic calculation that the phase-conjugation process is inherently noisy. We show that, for any state of the incident field, the statistical properties of the reflected field can always be described by a classical distribution function regardless of the state of the input field and that the reflected field possesses super-Poissonian phonon statistics. When an aberrator, which we model as a lossless scatterer, is placed in front of the PCM, additional fluctuations can arise because of the coupling of vacuum modes to the reconstructed-field modes.

We first consider the case in which a single-mode field is incident on a phase-conjugate mirror. We represent this field in the scalar approximation as

$$\hat{E}_i(\mathbf{r},t) = C\hat{a}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{H.a.} \tag{1}$$

where $\hat{a}$ denotes the photon annihilation operator of the incident-field mode and $C = -i(\hbar\omega/2\epsilon_0 V)^{1/2}$ where ϵ_0 is the permittivity of free space and V is the quantization volume. The field leaving the PCM is similarly represented as

$$\hat{E}_r(\mathbf{r},t) = C^*\hat{b}\, e^{i(-\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{H.a.} \tag{2}$$

where $\hat{b}$ represents the annihilation operator for the reflected mode. As photon operators, $\hat{a}$ and $\hat{b}$ and their Hermitean adjoints $\hat{a}^\dagger$ and $\hat{b}^\dagger$ must obey the canonical commutation relations[5] $[\hat{a},\hat{a}^\dagger] = [\hat{b},\hat{b}^\dagger] = 1$. In the classical description of the phase-conjugation process, the operators $\hat{a}$ and $\hat{b}$ are replaced by c-numbers that are related by $b = va^*$, where v denotes the amplitude reflectivity of the PCM. However, direct calculation shows that in the quantum-mechanical treatment the phase-conjugation process cannot be described solely through the relation $\hat{b} = v\hat{a}^\dagger$, since this relationship is incompatible with the commutation relations. In order to maintain the commutation relations, we postulate that the photon operators are related through[5-7]

$$\hat{b} = v\hat{a}^\dagger + \hat{L}, \tag{3}$$

where $\hat{L}$ represents a Langevin noise operator that obeys the commutation relation $[\hat{L},\hat{L}^\dagger] = |v|^2 + 1$ and satisfies the conditions $\langle\hat{L}\rangle = \langle\hat{L}^\dagger\rangle = 0$. For the case of a PCM based on degenerate four-wave mixing,[8,9] the Langevin noise operator can be identified with the amplified vacuum mode entering through the open rear port of the four-wave mixer. As in the case of degenerate four-wave mixing, we also assume that the expectation value of any normally ordered product of powers of $\hat{L}$ and $\hat{L}^\dagger$ is zero and that $\hat{L}$ and $\hat{a}$ are uncorrelated. The present treatment based on considerations of commutation relations of the optical field shows that the presence of this noise source is required for *any* PCM regardless of the physical mechanism leading to phase conjugation.

We now use Eq. (3) to calculate the statistical properties of the field generated by the PCM. We use the diagonal coherent-state representation[10,11] for the density operator $\hat{\rho}_b$ of the reflected field mode:

$$\hat{\rho}_b = \int \phi_b(\beta)\, |\beta\rangle\langle\beta|\, d^2\beta, \tag{4}$$

where $\phi_b(\beta)$ is the phase-space density. The phase-space density is a useful indicator of whether the state of the field is classical or quantum mechanical in nature[12]: When $\phi_b(\beta)$ does not behave like a classical probability density [i.e., when $\phi_b(\beta)<0$], the field mode $\hat{b}$ is inherently quantum mechanical in character and has no classical counterpart. We express $\phi_b(\beta)$ as a complex Fourier transform of the normally ordered characteristic function $\chi_b^{(N)}(\eta)$,[13]

$$\phi_b(\beta)=\pi^{-2}\int \chi_b^{(N)}(\eta)e^{\eta^*\beta-\beta^*\eta}d^2\eta, \tag{5}$$

where $\chi_b^{(N)}(\eta)$ is given by

$$\chi_b^{(N)}(\eta)=\langle e^{\hat{b}^\dagger\eta}e^{-\eta^*\hat{b}}\rangle. \tag{6}$$

We express $\phi_b(\beta)$ in terms of the phase-space density $\phi_a(\alpha)$ of the incident-field mode $\hat{a}$ by substituting Eq. (3) for $\hat{b}$ in Eq. (5), normal ordering the expression in $\hat{a}$, and using the optical equivalence theorem[10] to give the following equation:

$$\phi_b(\beta)=|\pi\nu|^{-2}\int \phi_a(\alpha)e^{-|(\beta/\nu)-\alpha^*|^2}d^2\alpha. \tag{7}$$

The right-hand side of Eq. (7) can be shown to be equal to the matrix element $\langle(\beta/\nu)^*|\hat{\rho}_a|(\beta/\nu)^*\rangle/\pi|\nu|^2$, which must always be positive semidefinite since $\hat{\rho}_a$ is a positive semidefinite operator. This result shows that even for an incident field which may be in a pure quantum state (e.g., amplitude or quadrature squeezed) the phase-conjugate reflected field can always be described classically. Hence, the desirable features of the initial state, such as reduced quantum noise, will be lost in the phase-conjugate process. This behavior is unlike that of a normal phase-insensitive amplifier for which the amplified field retains part of the quantum-mechanical nature of the input field up to a particular value of the gain.[14] Our conclusion that the output light can always be described classically holds whenever the Langevin operator $\hat{L}$ possesses the properties described in the discussion following Eq. (3). For the case of a general four-wave mixer,[8] the generated field can possess nonclassical features if the field injected through the rear input port is nonclassical.

We now give two examples of particular states of the input field and the resulting phase-space density for the reflected field. We first consider the case in which the input field is in the coherent state $|\alpha'\rangle$, and consequently the phase-space density is $\phi_a(\alpha)=\delta^2(\alpha-\alpha')$. The integral in Eq. (7) then trivially reduces to the expression

$$\phi_b^C(\beta)=|\pi\nu|^{-2}\exp[-|(\beta-\nu\alpha'^*)/\nu|^2]. \tag{8}$$

This distribution is similar to that which describes a thermal light source, but is centered on an amplitude with an expected value of $\beta=\nu\alpha'^*$ and has a characteristic spread equal to $|\nu|$. The next case we treat is one in which the incident field is in a Fock state $|n\rangle$. The phase-space density of this incident field has properties unlike that of a probability density in that it contains nth-order derivatives of a δ function.[12] Nevertheless, the phase-space density of the conjugate field is given by the expression

$$\phi_b^F(\beta)=(\pi|\nu|^2 n!)^{-1}|\beta/\nu|^{2n}e^{-|\beta/\nu|^2}, \tag{9}$$

which has the form of a probability density. Thus, the statistical properties of the reflected field can be described classically.

We have also calculated the statistical fluctuations in the number of photons contained in a mode of the reflected field. Through use of Eq. (3) [or Eq. (7)] we calculate $\langle\hat{n}_b\rangle=\langle\hat{b}^\dagger\hat{b}\rangle$ and $\langle\Delta\hat{n}_b^2\rangle=\langle\hat{n}_b^2\rangle-\langle\hat{n}_b\rangle^2$ and find that the reflected-field mode always possesses super-Poissonian statistics ($\langle\Delta\hat{n}_b^2\rangle>\langle\hat{n}_b\rangle$) even for input fields that are sub-Poissonian ($\langle\Delta\hat{n}_a^2\rangle<\langle\hat{n}_a\rangle$). The nature of this increase in fluctuations can be understood by noting that, according to Eq. (3), $\hat{a}^\dagger$ and not $\hat{a}$ has effectively become the relevant dynamical variable of the reflected field. Mandel[15] has shown that any quantum detector that operates by means of stimulated emission (i.e., by means of $\hat{a}^\dagger$) will be noisier than a detector that operates by the more conventional method based on the absorption of photons (i.e., by mean of $\hat{a}$). This noise can also be considered to be a manifestation of the noise inherent in all linear amplifiers[7,16] including those that are phase conjugating.[6,17]

We next consider the case in which an aberrating medium, which we model as a lossless scatterer, is placed in front of the PCM. We consider the extent to which the phase-conjugation process is capable of removing the aberrations from the conjugate field after its return passage through the aberrator. The optical field is described in terms of operators $\hat{a}_j$ ($\hat{a}_i'$) and $\hat{b}_i$ ($\hat{b}_j'$), shown in Fig. 1, which respectively represent the forward- and backward-traveling field modes on the left (right) of the scatterer. We treat the lossless scatterer as a $2N$-port network[18] which couples the $2N$ spatial modes on each side of the scatterer. The scattering process can then be described by the following equations:

$$\hat{a}_i'=\sum_{j=1}^{N}t_{ij}\hat{a}_j+\sum_{j=1}^{N}r_{ij}\hat{b}_j' \tag{10a}$$

and

$$\hat{b}_i=\sum_{j=1}^{N}t_{ij}'\hat{b}_j'+\sum_{j=1}^{N}r_{ij}'\hat{a}_j. \tag{10b}$$

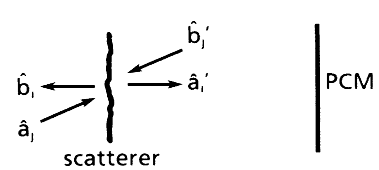

FIG. 1. An aberrating medium modeled as a lossless scatterer is placed in front of a PCM.

Here t_{ij} (t'_{ij}) describes the forward scattering of the jth mode on the left-hand (right-hand) side of the scatterer into the ith mode on the right-hand (left-hand) side of the scatterer. Likewise, r_{ij} (r'_{ij}) describes the backward scattering of mode j into mode i on the left-hand (right-hand) side of the scatterer. We next adopt the vector notation[19] in which $\hat{\mathbf{a}}$, $\hat{\mathbf{a}}^\dagger$, $\hat{\mathbf{a}}'$, $\hat{\mathbf{a}}'^\dagger$, $\hat{\mathbf{b}}$, $\hat{\mathbf{b}}^\dagger$, $\hat{\mathbf{b}}'$, and $\hat{\mathbf{b}}'^\dagger$ represent column vectors whose ith elements are $\hat{a}_i$, $\hat{a}_i^\dagger$, $\hat{a}'_i$, $\hat{a}_i'^\dagger$, $\hat{b}_i$, $\hat{b}_i^\dagger$, $\hat{b}'_i$, and $\hat{b}_i'^\dagger$, respectively. We can then express Eqs. (10) in the compact matrix form,

$$\begin{bmatrix}\hat{\mathbf{a}}'\\ \hat{\mathbf{b}}\end{bmatrix}=\begin{bmatrix}\mathbf{T} & \mathbf{R}\\ \mathbf{R}' & \mathbf{T}'\end{bmatrix}\begin{bmatrix}\hat{\mathbf{a}}\\ \hat{\mathbf{b}}'\end{bmatrix}=\mathbf{S}\begin{bmatrix}\hat{\mathbf{a}}\\ \hat{\mathbf{b}}'\end{bmatrix}, \tag{11}$$

where $\mathbf{T}$, $\mathbf{T}'$, $\mathbf{R}$, and $\mathbf{R}'$ are $N\times N$ matrices whose matrix elements are given by t_{ij}, t'_{ij}, r_{ij}, r'_{ij}, respectively. The scattering matrix $\mathbf{S}$ must be unitary ($\mathbf{S}\mathbf{S}^\dagger=\mathbf{S}^\dagger\mathbf{S}=\mathbf{I}_{2N}$, where $\mathbf{I}_{2N}$ is the $2N\times 2N$ identity matrix) since we have assumed that the aberrator is lossless. We also assume that the scatterer is symmetric such that the reciprocity conditions become $\mathbf{T}'=\mathbf{T}^{\mathrm{T}}$, $\mathbf{R}=\mathbf{R}^{\mathrm{T}}$, and $\mathbf{R}'=\mathbf{R}'^{\mathrm{T}}$, where superscript T denotes transpose. From these conditions of unitarity and reciprocity, we derive the following expressions which completely specify the relationships between the t_{ij}, t'_{ij}, r_{ij}, and r'_{ij}:

$$\mathbf{T}\mathbf{T}^\dagger+\mathbf{R}\mathbf{R}^\dagger=\mathbf{I}_N, \tag{12a}$$

$$\mathbf{T}'\mathbf{T}'^\dagger+\mathbf{R}'\mathbf{R}'^\dagger=\mathbf{I}_N, \tag{12b}$$

and

$$\mathbf{T}\mathbf{R}'^\dagger+\mathbf{R}\mathbf{T}'^\dagger=0. \tag{12c}$$

We apply Eq. (3) to each mode such that $\hat{\mathbf{b}}'=\nu\hat{\mathbf{a}}'^\dagger+\hat{\mathbf{L}}$ and use Eq. (11) to solve for the reconstructed field $\hat{\mathbf{b}}$ in terms of the input field operators and the PCM noise operators:

$$\hat{\mathbf{b}}=\mathbf{T}'[\mathbf{I}_N-|\nu|^2\mathbf{R}^*\mathbf{R}]^{-1}[\nu\mathbf{T}^*\hat{\mathbf{a}}^\dagger+|\nu|^2\mathbf{R}^*\mathbf{T}\hat{\mathbf{a}}+\nu\mathbf{R}^*\hat{\mathbf{L}}^\dagger+\hat{\mathbf{L}}]+\mathbf{R}'\hat{\mathbf{a}}. \tag{13}$$

We see from this matrix equation that each element $\hat{b}_i$ in general contains not only a term proportional to $\hat{a}_i^\dagger$, as would be desired for ideal phase conjugation, but also terms like $\hat{a}_j^\dagger$ (for all $j\neq i$) and $\hat{a}_j$ (for all j). It can be seen by inspection that for the case in which only a single mode i is excited at the input, the presence of the second term in Eq. (13) degrades the quality of the phase-conjugate signal, in that it leads to an output whose expectation value depends in part upon $\langle\hat{a}_i\rangle$ and not solely upon $\langle\hat{a}_i^\dagger\rangle$. The other terms mentioned above can lead to an increase in fluctuations in the reconstructed-field mode $\hat{b}_i$. However, there are two cases in which $\hat{b}_i$ is simply proportional to $\hat{a}_i^\dagger$ plus noise terms due solely to the PCM. The first case is when there is no backscattering ($r_{ij}=r'_{ij}=0$ for all i and j) so that Eq. (13) reduces to $\hat{\mathbf{b}}=\nu\hat{\mathbf{a}}^\dagger+\hat{\mathbf{L}}$ and all the effects of the scatterer have been removed. The second case occurs when $|\nu|=1$, so that Eq. (13) simplifies to

$$\hat{\mathbf{b}}=\hat{\mathbf{a}}^\dagger+(\mathbf{T}^*)^{-1}(\hat{\mathbf{L}}+\mathbf{R}^*e^{i\theta}\hat{\mathbf{L}}^\dagger), \tag{14}$$

where $\theta=\arg\nu$. Thus, the condition that either $|\nu|=1$ or $r_{ij}=r'_{ij}=0$, which within the context of the classical theory implies that the phase conjugation occurs with highest fidelity, is also the condition which within the context of quantum theory implies that the scatterer does not couple vacuum fluctuations into the reconstructed-field mode. Our conclusions regarding fundamental limitations to the aberration-correcting ability of phase conjugation are somewhat different from those reached by Band, Heller, and Kafri,[20] who conclude that the scatterer does introduce noise into the reconstructed beam. Their calculation differs from ours in that they do not explicitly treat the dynamics of the PCM and, hence, do not include the effects of noise introduced by the PCM. However, they argue on the basis of thermodynamic reasoning that the entropy of the light field must increase upon transit through an aberrator, and, since the entropy cannot decrease in the second pass, the reconstructed field will be imperfect in the sense that a certain minimum number of photons will be scattered into other field modes. In contrast, our calculation assumes a deterministic model for the aberrator, and for this reason we have not included entropy considerations into our treatment.

In the treatment presented here, we assume that the PCM is ideal in the sense that all spatial modes are reflected with the same phase-conjugate reflectivity ν. When ν is not constant, perfect reconstruction of the field cannot occur even in the limit of no backscattering, and thus vacuum modes will then couple to the reconstructed initially populated modes leading to additional fluctuations. We have considered only single-frequency operation of the PCM, but in practice the PCM will be characterized by some nonzero bandwidth and ν will be frequency dependent. Since phase conjugation couples modes in pairs (symmetrically displaced about the pump frequency of the PCM), our treatment can be easily extended to include these modes by the introduction of a frequency-dependent ν. Even for the case in which the applied input field is at a single frequency, the nonzero bandwidth of the PCM implies that the field leaving the PCM will contain noise photons at the other frequencies which will contribute additional background noise.

In conclusion, we have shown that quantum noise imposes a fundamental limitation to the performance of PCM's, especially when used with very weak input fields. Our results potentially have important implications for the use of phase conjugation for incident fields in nonclassical states, for the noise properties of the radiation produced by a phase-conjugate resonator,[21] for the use of phase conjugation for information processing,[22] and for the dynamics of atoms located in front of PCM's.[23]

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