Critical Behavior in (2+1)-Dimensional QED

Thomas Appelquist and Daniel Nash
Physics Department, Yale University, New Haven, Connecticut 06511

and

L. C. R. Wijewardhana
Physics Department, University of Cincinnati, Cincinnati, Ohio 45221

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QED in 2+1 dimensions, analyzed in the $1/N$ expansion, is shown to exhibit a critical behavior as the number N of fermions approaches $32/\pi^2$. The dynamical mass has a universal scaling behavior at the critical point. The existence of criticality is discovered first by an analytic study of the gap equation and the effective potential. Its existence and universality is then confirmed numerically. Similarities to four-dimensional theories are discussed.

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Quantum electrodynamics in 2+1 dimensions (QED3) is a superrenormalizable gauge theory with some resemblance to four-dimensional theories when analyzed in a $1/N$ expansion.[1] In the massless theory, the dimensionful coupling $\alpha = Ne^2/8$, which is kept fixed when the number of fermions N is taken to infinity, provides the only fixed scale in the theory. At momentum scales p beyond α, the theory is rapidly damped. For $p \ll \alpha$, the singular behavior of the loop expansion is softened in the $1/N$ expansion. To each order in this expansion, the Green's functions of the massless theory can be shown to be infrared finite.[2,3] The infrared finiteness is a consequence of an effective low-energy scale invariance that the theory exhibits to each order in $1/N$.[3] The effective interaction strength in this limit is proportional to $1/N$.

Despite this exemplary behavior of the massless theory, dynamical symmetry breaking leading to a nonzero fermion mass could take place. This question has been examined previously[4,5] to first and second order in the $1/N$ expansion. Incomplete analytic studies indicated that dynamical fermion mass generation takes place for arbitrarily large N.[4,5] Numerical studies also demonstrated mass generation, but were restricted to $N<3$ in order to achieve convergence of the iterative procedure. It is important to settle this question. Because of its low-energy scale invariance, QED3 is a useful theoretical laboratory to gain insight into dynamical symmetry breaking in four-dimensional theories.

In this paper, it will be shown that to lowest order in the $1/N$ expansion of the Dyson-Schwinger kernel, dynamical mass generation takes place only for $N<N_c = 32/\pi^2$.[6] To what extent this result is modified by higher-order corrections remains an open question. Some concluding remarks will address this issue.

The Lagrangean of QED3 is

$$L = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi - \tfrac{1}{4}F_{\mu\nu}^2, \tag{1}$$

where ψ is taken to be a four-component complex spinor. With the four-component notation, 4×4 matrices γ_3 and γ_5 that anticommute with γ_0, γ_1, and γ_2 can be introduced. Together with Eq. (1) and $[\gamma_3, \gamma_5]$, they generate a U(2) symmetry for each spinor. Thus the full global "chiral" symmetry of the Lagrangean (1) is $U(2N)$ where N is the number of four-component spinors. A mass term $m\bar{\psi}\psi$ would break this symmetry to $U(N)\otimes U(N)$. The dynamical generation of such a mass will be considered here. It is also possible to include a chiral-invariant but parity-nonconserving mass in QED3. It has been argued,[7] however, that the dynamical generation of the parity-invariant mass is energetically preferred to the parity-nonconserving one. In this Letter, we shall consider only the parity-conserving case.

The inverse fermion propagator is $S(p)^{-1} = -p\!\!\!/A(p) + \Sigma(p)$, where $\Sigma(p)$ is a dynamical, parity-conserving mass taken to be the same for all the fermions, and $A(p)$ is wave-function renormalization. The lowest-order gap equation for $\Sigma(p)$ is formed by our setting $A=1$ and neglecting other higher-order corrections. It is

$$\Sigma(p) = \frac{8\alpha}{N}\int \frac{d^3k}{(2\pi)^3}\frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\gamma^\nu}{k^2+\Sigma^2(k)}, \tag{2}$$

where, in the Landau gauge,

$$D_{\mu\nu}(p-k) = \frac{g_{\mu\nu} - (p-k)_\mu (p-k)_\nu/(p-k)^2}{(p-k)^2[1+\Pi(p-k)]}. \tag{3}$$

To leading order in $1/N$, $\Pi(p-k)$ comes from a single fermion loop. For $p,k \gg \Sigma(p)$, it can be explicitly evaluated to give $\Pi(p-k) = \alpha/|p-k|$. Angular integration in Eq. (2) then gives

$$\Sigma(p) = \frac{4\alpha}{\pi^2 Np}\int_0^\infty dk \frac{k\Sigma(k)}{k^2+\Sigma^2(k)} \ln\left(\frac{k+p+\alpha}{|k-p|+\alpha}\right). \tag{4}$$

It is clear from the above discussion that this form of the lowest-order gap equation does not correctly treat the nonlinear regime $k \leq \Sigma(k)$. For $\Sigma(k) \ll \alpha$, however, as will be the case for N near N_c, this will not be important.

The integral equation (4) is rapidly damped for $p > \alpha$. For $p < \alpha$, it takes the approximate form

$$\Sigma(p) = \frac{4}{\pi^2 N p} \int_0^\alpha dk \frac{k\Sigma(k)}{k^2+\Sigma^2(k)} (k+p-|k-p|). \tag{5}$$

This approximate equation, in which the logarithm of Eq. (4) is replaced by a hard cutoff at $p=\alpha$, will first be solved analytically to provide insight into the critical behavior. A numerical study of the full Eq. (4) will then reveal the same value for N_c and the same approach to criticality.

Equation (5) can be replaced by the differential equation

$$\frac{d}{dp}\left[p^2 \frac{d\Sigma(p)}{dp}\right] = -\left(\frac{8}{\pi^2 N}\right) \frac{p^2\Sigma(p)}{p^2+\Sigma^2(p)}, \tag{6}$$

together with the boundary conditions

$$0 \leq \Sigma(0) < \infty \tag{7a}$$

and

$$\left[p\frac{d\Sigma(p)}{dp} + \Sigma(p)\right]_{p=\alpha} = 0. \tag{7b}$$

Equation (7b) insures the absence of a bare mass and the conservation of the axial vector current. If $\Sigma(p) \ll \alpha$, there will exist a regime $p \gg \Sigma(p)$ where Eq. (6) can be linearized. We first assume this to be the case and then justify it in the limit $N \to 32/\pi^2$. In the linear regime, the solution to the differential equation is

$$\Sigma(p) \sim p^a \tag{8}$$

where

$$a = -\tfrac{1}{2} \pm \tfrac{1}{2}(1-32/\pi^2 N)^{1/2}. \tag{9}$$

The system of equations (6)–(9) has a precise counterpart in quenched QED4 solved in ladder approximation. There the dimensionless gauge coupling replaces $1/N$ and an ultraviolet cutoff replaces α. We simply carry over the well-known analysis of this theory[8,9] and report the corresponding results for QED3.

For $N > 32/\pi^2$, the boundary condition (7a) leads to the dominance of the solution $a = -\frac{1}{2} + \frac{1}{2}(1-32/\pi^2 N)^{1/2}$ in the linear regime $p \gg \Sigma(p)$. This behavior is incompatible with the boundary condition (7b) and therefore $\Sigma(p)=0$ is the only solution. For $N < 32/\pi^2$, the linearized equation has an oscillatory solution which can be written in the form

$$\Sigma(p) = p^{-1/2} \sin\left(\tfrac{1}{2}(32/\pi^2 N - 1)^{1/2}\{\ln[p/\Sigma(0)]+\delta\}\right), \tag{10}$$

where δ is a phase and $\Sigma(0)$ has been used to scale the logarithm. The ultraviolet boundary condition (7b) in the limit $N \to 32/\pi^2$ then gives

$$\tfrac{1}{2}(32/\pi^2 N - 1)^{1/2}\{\ln[\alpha/\Sigma(0)]+\delta\} = n\pi - (32/\pi^2 N - 1)^{1/2}, \tag{11}$$

where $n = 1,2,3,\ldots$. Thus

$$\Sigma(0) = \alpha e^{(\delta+2)} \exp\left[\frac{-2n\pi}{(32/\pi^2 N - 1)^{1/2}}\right]. \tag{12}$$

The phase δ can depend on N but is not expected to be singular in the limit $N \to 32/\pi^2$. The solution $n=1$ can be shown to give the lowest vacuum energy and is therefore the vacuum solution. It gives the largest dynamical mass $\Sigma(0)$ and a monotonically decreasing (nodeless) behavior of $\Sigma(p)$ for $0 \leq p \leq \alpha$.

Equation (12), with $n=1$, exhibits the critical behavior. $\Sigma(0)$ vanishes as $N \to 32/\pi^2$ from below, as described by the factor $\exp[-2\pi/(32/\pi^2 N - 1)^{1/2}]$. It is this factor that will be shown by a numerical study to be universal in the sense of being independent of the ultraviolet behavior $p \geq \alpha$. In particular, the same factor will appear in the solution to the full Eq. (4) in which the high-momentum components are cut off softly rather than abruptly as in the approximate Eq. (5). The numerical study will also demonstrate that this factor is insensitive to the nonlinear regime $p \leq \Sigma(0)$. The detailed connection between the linear regime and $p=0$ is contained in the phase δ in Eq. (10). It will be shown that the factor e^δ in Eq. (12) is nonsingular as $N \to 32/\pi^2$, independently of the treatment of the nonlinear regime $p \leq \Sigma(0)$. This is important since, as pointed out above, Eq. (4) does not treat this regime correctly. What is happening is that as $N \to 32/\pi^2$, the range between $\Sigma(0)$ and α increases exponentially. The infrared regime $p \leq \Sigma(0)$ and the ultraviolet regime $p \geq \alpha$ are not increasing in this way and become relatively less important. The universal factor $\exp[-2\pi/(32/\pi^2 N - 1)^{1/2}]$ arises from the growing intermediate regime.

Before we summarize the numerical study it is also worth our mentioning that an independent derivation of $N_c = 32/\pi^2$ can be given by considering the local stability of the effective potential about $\Sigma = 0$.[10] For $N > N_c$, the $\Sigma = 0$ configuration is locally stable while for $N < N_c$ it is unstable. This parallels a derivation of the critical coupling for quenched QED4.[11]

In the numerical study, solutions to Eq. (4) are found by an iterative procedure. As an initial guess, $\Sigma(p)$ is taken to be a constant somewhat above the anticipated $\Sigma(0)$. The numerical integration is cut off in the ultraviolet, initially at some $\Lambda \gg \alpha$. Since the theory is rapidly damped beyond α, nothing is sensitive to this Λ. For a series of values of N below $32/\pi^2$, the iterative procedure converges to give a finite $\Sigma(p)$. For $N > 32/\pi^2$, however, the iterations converge to zero. For $N < 32/\pi^2$, $\Sigma(p)$ is a monotonically decreasing function of p. In Fig. 1, $-\ln[\Sigma(0)/\alpha]$ is plotted versus N, showing evidence for a singularity as $N \to 32/\pi^2$. In Fig. 2, the plot of $-\ln[\Sigma(0)/\alpha]$ vs $1/(32/\pi^2 N - 1)^{1/2}$ shows the expected straight-line behavior with a slope approaching 2π. The evidence from the numerical study is that Eq. (12) correctly describes the solution to Eq. (4) as $N \to 32/\pi^2$ from below. In particular, the factor e^{δ} exhibits very little sensitivity to N in this limit. A surprising feature of Fig. 2 is that Eq. (12) provides a good fit for N as low as unity.

The sensitivity of the solution to the details of the ultraviolet cutoff can easily be checked numerically. A simple procedure is to change the computer cutoff Λ relative to the intrinsic cutoff α. We have rerun the analysis for $\Lambda = O(\alpha)$ as well as for $\Lambda \ll \alpha$. The latter case should be approximated by Eq. (5) with α replaced by Λ. Numerical results for $\Lambda \ll \alpha$ are also shown in Figs. 1 and 2. The general result is that for any choice of Λ, $\Sigma(p)$ is nonzero and monotonically decreasing when $N < 32/\pi^2$. In the limit $N \to 32/\pi^2$, with α and Λ fixed, $\Sigma(0)$ vanishes according to

$$\Sigma(0) = f\left[\frac{\alpha}{\Lambda}\right] \alpha \exp\left[\frac{-2\pi}{(32/\pi^2 N - 1)^{1/2}}\right], \tag{13}$$

where f is some dimensionless function of the ratio α/Λ. The exponential factor appears to be a universal quantity, insensitive to the above changes in the ultraviolet regime.

Insensitivity of this factor to the details of the nonlinear regime, where Eq. (4) is not correct in detail, can also be checked numerically. The integrand in Eq. (4) can be modified for $k \leq \Sigma(0)$ in essentially any way that is insensitive to N and that does not change the order of magnitude of the integral. This has been done in a variety of ways. In each case it is found in agreement with the arguments presented above, that the behavior in the limit $N \to 32/\pi^2$ is described by the exponential factor in Eq. (13).

Since spontaneous chiral symmetry breaking takes

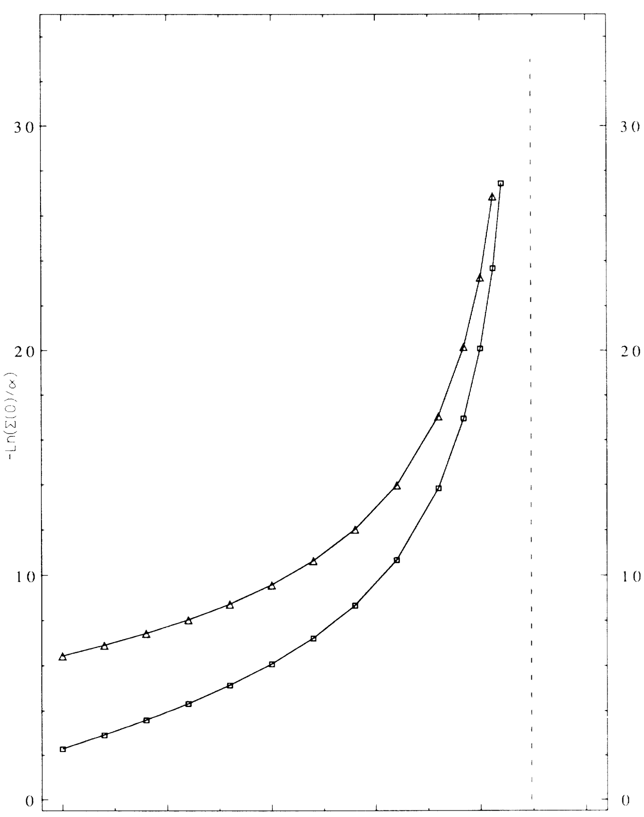

FIG. 1. $-\ln[\Sigma(0)/\alpha]$ vs N. The squares correspond to $\Lambda \gg \alpha$, and the triangles to $\Lambda \ll \alpha$, where Λ is the computer cutoff. Both curves appear to approach an asymptote at $32/\pi^2$. Computational limits prevent checking $N > 3.1$.

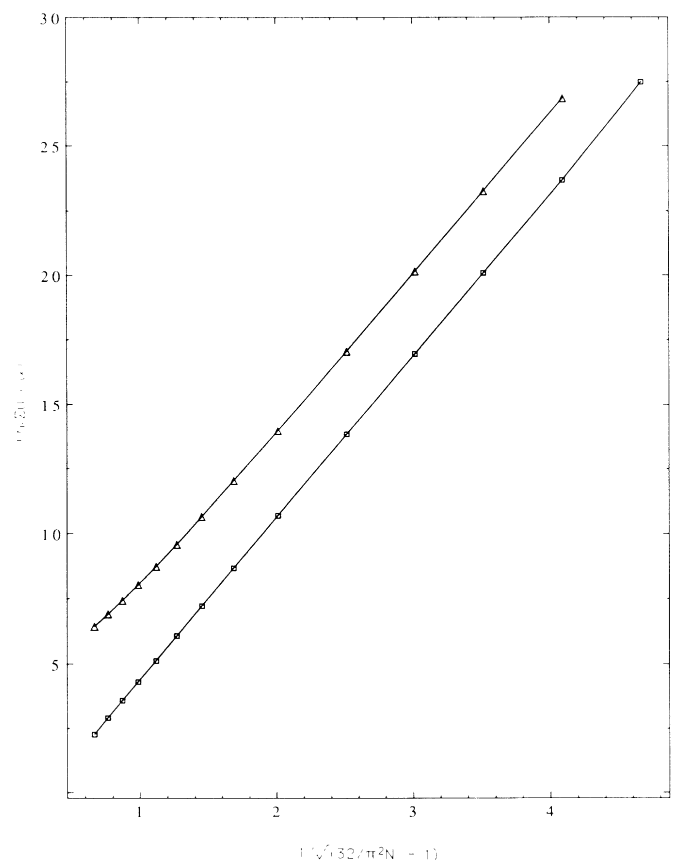

FIG. 2. $-\ln[\Sigma(0)/\alpha]$ vs $1/(32/\pi^2 N - 1)^{1/2}$. The squares correspond to $\Lambda \gg \alpha$, and the triangles to $\Lambda \ll \alpha$. As $N \to 32/\pi^2$ the slopes of both lines tend towards 2π. A lower-precision algorithm gave similar straight-line behavior, but with shallower slopes. Improvements in precision led to a series of lines, whose slopes appear to converge to 2π.

place only for $N < N_c = 32/\pi^2$, higher-order corrections to Eq. (4) might be expected to be large. The point has already been made that to determine N_c and the approach to criticality, the relevant momentum range is $\Sigma(0) \ll p \ll \alpha$. Thus, to estimate higher-order corrections to N_c and to the exponential factor in Eq. (18), it should suffice to compute the higher-order corrections to the linearized gap equation in the limit $\alpha \to \infty$:

$$\Sigma(p) = \frac{8}{\pi^2 N p} \int_0^\infty \frac{dk}{k} \Sigma(k) \left[\frac{k+p-|k-p|}{2} \right]. \quad (14)$$

This equation leads to Eq. (8) with $a(1+a) = -\frac{1}{4} \times 32/\pi^2 N$ and therefore to Eq. (9).

The higher-order corrections will include the wave-function renormalization $A(k)$, as well as the vertex correction and higher-order corrections to the gauge boson propagator. There is also the crossed ladder contribution to the kernel. In Ref. 5, it was argued that these contributions will lead to $1/N$ (and higher) corrections to the integrand of Eq. (14) with no dynamical enhancements. If that is the case, a corrected expression for $a(a+1)$ should emerge from the computation. It will read

$$a(1+a) = -\tfrac{1}{4}(32/\pi^2 N)(1 + c32/\pi^2 N + \cdots). \quad (15)$$

The corrected value of N_c will presumably be determined by our setting $(32/\pi^2 N)(1 + c32/\pi^2 N + \cdots)$ equal to unity. This quantity will then also replace $32/\pi^2 N$ in the exponential factor in Eqs. (12) and (13).

The importance of the second-order term will depend on the size of the coefficient c. If it is considerably smaller than unity, for example, the second-order term will not change the qualitative features uncovered with the lowest-order kernel. This computation is currently under way.

To conclude, quantum electrodynamics has been shown to exhibit spontaneous chiral symmetry breaking when analyzed in a $1/N$ expansion, provided $N < 32/\pi^2$. This result has been established with only the first term retained in the $1/N$ expansion of the kernel of the gap equation and effective potential. Whether it is modified by higher-order corrections is currently under study. In the first-order analysis, the dynamical fermion mass vanishes as $N \to 32/\pi^2$ in a universal manner, independent of the details of the ultraviolet cutoff or the infrared, nonlinear regime.

If the result demonstrated here is not qualitatively changed by higher-order terms, it opens up a variety of questions. Are the broken-symmetry solutions local minima of the effective action? Is there a critical behavior for a parity-nonconserving mass? How is the critical behavior modified by finite temperature and the inclusion of supersymmetry? A lattice study of QED3 could help to shed light on all these questions.

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